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Multenions and Differential Invariants.

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§ 1. *Preliminary Explanations.*—Since multenions appear almost to have been designed by Nature to serve as an algebra dealing with such matters as differential invariants and relativity, I have thought it desirable at the present moment to present a summary of their properties. Towards the end I propose to enter into more detail in applying them to differential invariants, and to point out some of the special properties of multenions when n, the vector complexity of the system, has the particular value 4.

Clifford first, I believe, applied the subject to elliptic space of three dimensions. His bi-quaternion $q+\omega r$, where $\omega^2 = 1$, is the general member of a sub-algebra of multenions when $n = 4$. In the form of multenions suitable to deal with real questions of Relativity Hamilton's bi-quaternion $q+r\sqrt{(-1)}$ presents itself in a precisely similar manner, and of this I shall have more to say towards the end of the paper. Joly's 'Manual of Quaternions' has its last chapter devoted wholly to multenions. I believe the most elaborate attempt at development is my own paper in 'Proc. Roy. Soc., Edin.,' 1907–8, p. 503.

The notation used below is based as closely as possible on quaternion notation. It is practically identical with Joly's so far as the latter goes. It differs in several respects from the notation of my own former paper. The changes are all in the direction of simplicity, and back towards quaternion rules. The same applies to changes in terminology. These changes, which I believe are all for the better, are in the main due to friendly advice received from Prof. Knott in connection with the former paper.

Law I. Besides the unit scalar 1 there are given n generators, which will generally be regarded as representing n unit mutually orthogonal vectors in a

Euclidian space of n dimensions, $i_1, i_2, \ldots i_n$. These are such that all the ordinary laws of algebra except the commutative law of multiplication apply to the general integral expression q, called a multenion, where

$$q = \Sigma\,(x_0+x_1i_1+x_2i_2+\ldots)(x_0'+x_1'i_1+x_2'i_2+\ldots)\ldots, \tag{1}$$

where every x is a scalar. Since any x may be negative we understand this statement to include the ordinary algebraic uses of the signs $+$ and $-$; but we do not understand that any assertion has been made about the use of the sign $\div$ [$q\div r$ can later be understood to be an alternative for qr^{-1}].

Law II. Scalars are commutative not only with each other but with each of the n primitive unit vectors. [Thus scalars are commutative with multenions.]

Law III. [$i_1{}^2$ means i_1i_1, etc.]

$$-1 = i_1{}^2 = i_2{}^2 = \ldots = i_n{}^2.$$

Law IV. In a product, called a primitive vectorium, $i_ai_b\ldots$, of primitive unit vectors, if two adjacent *different* vectors be interchanged, the product merely changes sign.

Such are the assumptions. The rest is development, aided of course by suitable terminology and symbolisation.

Def. 1. A primitive unit means any one of the following: (1) the scalar 1, (2) any primitive unit vector i_a, (3) any other primitive vectorium $i_ai_b\ldots$. [Strictly (3) includes (2) and in a sense (1).]

Def. 2. An expression of the form $\rho = \sum\limits_{c=1}^{n} x_ci_c$ where each x is a scalar, is called a vector, or a V_1 or a ${}_nV_1$,

Def. 3. Any multenion is obviously capable of expression in the form $\Sigma xi_ai_b\ldots i_k$ where $i_a, i_b, \ldots i_k$ are m different primitive vectors and m ranges from 0 to n, m being given V_mq means the sum of all parts of the multenion for that value of m. Thus

$$q = V_0q+V_1q+V_2q+\ldots+V_nq.$$

V_aq is said to be the part of q of homogeneity a and is called a V_a or a ${}_nV_a$ or a homogeneous multenion or a hyper vector. [These alternatives are intended to invite suggestions for a settlement of the name. I should have preferred "an a-vector" but this clashes with relativity meanings of 4-vector and 6-vector. In multenions with $n=4$, there are two kinds of 4-vector ${}_4V_1$ and ${}_4V_3$.]

V_oq will generally in words be called the scalar part of q but in the present connection we must not use Sq as an alternative for V_0q. In applications we very decidedly need to be able, in one system of multenions ($n=4$) to make

no change in the quaternion notation but yet to use both the quaternion notation and the full multenion notation as applying to one symbol q. Thus

$$Sq = V_0 q + V_4 q, \qquad Vq = V_2 q.$$

Endless confusion would result from allowing V_0 to be represented by S.

Def. 4. If $\alpha_1, \alpha_2, \ldots \alpha_a$ are any a-vectors then $V_a \alpha_1 \alpha_2 \ldots \alpha_a$ is called a vectorium. [Hence we have already called such an expression as $i_1 i_2 i_3$ a primitive vectorium.]

Def. 5. A linear multenion function of a multenion is called a multenion linity. Similarly for a vector linity or a ${}_n V_a$ linity, etc. An extended vector linity is a special kind of multenion linity, Φ, defined from a given vector linity, ϕ. Its fundamental property is this: If $\alpha_1, \alpha_2, \ldots \alpha_a$ are any a-vectors whatever, a itself ranging from 0 to n, then

$$\Phi V_a \alpha_1 \alpha_2 \ldots \alpha_a = V_a \phi\alpha_1 \phi\alpha_2 \ldots \phi\alpha_a. \tag{2}$$

[It is not obvious that this remark covers a possibility in general. The definition, which will be modified below, is given here as a hint of the importance of vectoriums and extended linities in our subject.]

Usual Notations.

Scalars (V_0): a, b, c, g, h, k, l, m, n, t (= time), x, y, z, J ($= \sqrt{[-1]}$), θ (= arc), π.

General multenions: p, q, r, s.

Hyper vectors (homog. mult.) and vectoriums:

Homogeneities	V_a a	V_b b	V_c c	V_n n	V_2 2
Hyper vectors	u	v	w	$x\grave{\upsilon}$	ω
Vectoriums	$\grave{\alpha}$	$\grave{\epsilon}$	$\grave{\eta}$	$x\grave{\upsilon}$	$\grave{\omega}$

(These conventions should be noted and remembered. They immensely simplify the reading of formulæ by their property of minimising number of suffixes, etc.)

Vectors (V_1): α, β. γ, δ, ϵ, ζ, ξ, η, λ, μ, ν, ρ, σ, τ.

Linear functions; ξ, η, λ, μ, ν, ϖ, υ, ϕ, χ, ψ.

Vectoriums: $\grave{\alpha}$, $\grave{\epsilon}$, $\grave{\eta}$, $\grave{\iota}$, $\grave{o}$, $\grave{\rho}$, $\grave{\upsilon}$, $\grave{\omega}$.

Differential symbols: δ, Δ, H, L, d, $d\rho$, $d\grave{\rho}_b$, $d\varsigma_a$, $d\alpha$, $d\dot{b}$.

Selective linities, etc.: K, P, Q, R, S, T, U, V.

Def. 6. Complexes of vectors, hypervectors, multenions, etc., are sufficiently illustrated by multenions. If $q_1, q_2, \ldots, q_a$ are a given multenions, then all multenions of the form $\sum_{c=1}^{a} x_c q_c$ are said to form a complex. If $q_1, q_2, \ldots q_a$

can be made linearly dependent on b and no fewer, then b is called the complexity of the complex. Thus the complexity of the complex of all vectors is n, of all multenions it is 2^n, of all multenions of homogeneity a it is ${}_nC_a$.

Def. 7. A unit vector or a unit vectorium is one, $\acute{a}$, such that $\acute{a}^2 = \pm 1$. [The square of every vectorium is a scalar, but u^2 is a scalar in general only in the four cases when $a = 0, 1, n-1$ or n.] Thus all the primitive vectoriums are unit vectoriums.

Def. 8. Two vectors, α, β, are said to be normal to one another when $V_0\alpha\beta = 0$. Two vector complexes are said to be completely normal to one another when every vector of the one complex is normal to every vector of the other.

Every one of the symbols V_a is a multenion linity. They are all commutative with one another. They are all idempotent, that is $V_a{}^2 = V_a$, and they all satisfy the equation $V_aV_b = 0$ when $a \neq b$.

There is another system of multenion linities of great utility and simplicity. They are 2^n in number. They are all commutative with each other and with each of the $n+1$ linities V_a. Let P_a be that definite linity of a general multenion, q, which simply reverses the sign of every i_a which occurs in the expression (1) for q. From the well-known properties of linities a product of the P's such as $P_1P_2P_3$ simply reverses the sign of every i_1, i_2, and i_3. These linities are (1) unipotent, that is, if

$$\phi = P_aP_b \dots P_k, \tag{3}$$

then $\phi^2 = 1$; (2) they are proscriptive, by which is meant that

$$\phi(qr) = \phi q\, \phi r, \tag{4}$$

where q and r are any two multenions. It is convenient to use P for the product $P_1P_2 \dots P_n$ of all the linities P_a, so that P reverses the sign of every primitive vector.

K is defined, consistently with its quaternion use, in the following manner. Kq is a linity of a general multenion, q, which reverses the sequence of the primitive unit vectors in every primitive vectorium $\acute{\iota}$, of q, *and also* reverses the sign of every primitive unit vector. Thus

$$K(i_1i_2i_3) = (-i_3)(-i_2)(-i_1) = i_1i_2i_3.$$

K is (1) unipotent, that is $K^2 = 1$; (2) it is commutative with every V and every P; (3) it is retroscriptive, that is $K(qr) = KrKq$. [To prove (3) first prove it when q and r are two primitive vectoriums and then generalise. Such a method of proof is very often applicable and may be referred to as proof by reduction to primitive units.]

But from these properties not only is

$$K(qr) = Kr \,.\, Kq, \tag{5}$$

but if ϕ be given by (4), $K\phi\,(=\phi K)$ is also retroscriptive. More generally, if $\phi, \psi, \chi, \ldots$, are any number of multenion linities, every one of which is either proscriptive or retroscriptive, then also is the product, $\phi, \psi, \chi, \ldots$, proscriptive or retroscriptive; it is proscriptive when the number of $\phi, \psi, \chi, \ldots$, which are retroscriptive is even, and it is retroscriptive when that number is odd.

The special retroscriptive PK (= KP) is of sufficient importance to have a symbol appropriated to it. We shall call it Q. Thus the product of any two of the three P, Q, K is equal to the third.

A caution to users of present methods in relativity and hyperbolic geometry seems desirable. It relates to the discussion of real questions by aid of the imagery

$$J = \sqrt{(-1)}. \tag{6}$$

Let us illustrate by anticipating our mode of treatment of real questions in relativity. We begin with the multenion system based on the primitive generators i_1, i_2, i_3, i_4. Then we put

$$i_4 = J\iota, \tag{7}$$

and drop out of most of our subsequent work all reference to the original i_4 and J_2, using ι instead with its property $\iota^2 = 1$, as opposed to $i_3{}^2 = -1$. Questions in relativity are then treated by aid of the multiplicative combinations of such expressions as

$$xi_1+yi_2+zi_3+ct\iota, \tag{8}$$

whereby, so long as all the scalars thereby introduced are real, we shall be dealing with real questions in relativity. The general multenion thus constructed contains 16 and not 32 independent real scalars. ι is for practical relativity purposes real. We shall call such a multenion system semi-real.

Now return to the meaning above assigned to K.

$$K(i_1i_2i_3i_4) = i_1i_2i_3i_4,$$

and also $$K(i_1i_2i_3\iota) = i_1i_2i_3\iota,$$

but whereas $$K(i_1i_2i_3i_4) = (i_1i_2i_3i_4)^{-1}, \tag{9}$$

it is not true that the corresponding equation holds when we replace i_4 by ι. I have therefore refrained from making the definition of K depend on such equations as (9). It is, nevertheless, a perfectly easy theorem to prove that

$$K\acute{\iota} = \acute{\iota}^{-1}, \tag{10}$$

when $\acute{\iota}$ is any product of our *original* generators $i_1, i_2, \ldots i_n$. This ceases to be true when ι takes the place of i_4, simply because J^{-1} is $-J$ and not $+J$.

It may be noted, however, that there is always a reproscriptive $K\phi = R$ to be found, which is such that for every semi-real vectorium $\acute{\iota}$, $R\acute{\iota} = \acute{\iota}^{-1}$.

Thus, suppose we replace $i_{a+1}, \dots i_n$ by $l_{a+1} \dots l_n$

where $\quad i_c = \mathrm{J}l_c$ and therefore $l_c^2 = 1, \qquad c = a+1, a+2, \dots n. \qquad (11)$

Then $$\mathrm{R} = \mathrm{K}\mathrm{P}_{a+1}\mathrm{P}_{a+2}\dots\mathrm{P}_n = \mathrm{Q}\mathrm{P}_1\mathrm{P}_2\dots\mathrm{P}_a, \tag{12}$$

is easily shown to have the required characteristic.

We now obviously have

$$\mathrm{P} = \mathrm{P}_1\mathrm{P}_2\dots\mathrm{P}_n, \tag{13}$$

$$\mathrm{P}u = (-)^a u, \qquad \mathrm{Q}u = (-)^{\frac{1}{2}a(a-1)}u, \qquad \mathrm{K}u = (-)^{\frac{1}{2}a(a+1)}u. \tag{14}$$

Further if we put

$$q_c = \mathrm{V}_c q, \qquad c = 0, 1, \dots n, \tag{15}$$

$$\left.\begin{aligned} \mathrm{P}q &= q_0 - q_1 + q_2 - q_3 + \dots \\ \mathrm{P}_{(0)}q &= \tfrac{1}{2}(1+\mathrm{P})q = q_0 + q_2 + q_4 + \dots \\ \mathrm{P}_{(1)}q &= \tfrac{1}{2}(1-\mathrm{P})q = q_1 + q_3 + q_5 + \dots \end{aligned}\right\}, \tag{16}$$

$$\mathrm{Q}q = q_0 + q_1 - q_2 - q_3 + q_4 + q_5 - \dots, \tag{17}$$

$$\mathrm{K}q = q_0 - q_1 - q_2 + q_3 + q_4 - q_5 - \dots, \tag{18}$$

Clearly, we might introduce Q_0, Q_1, and K_0, K_1, corresponding to $\mathrm{P}_{(0)}$ and $\mathrm{P}_{(1)}$.

All the linities recently considered (V_c, ϕ, $\mathrm{K}\phi$) may be referred to as the capital linities.

§ 2. *Theorem of Independence and Allied Algebraic Features.*— If $q_1q_2 = -q_2q_1$, then q_1 and q_2 are said to be anti-commutative. In the following enunciation, a product of a different multenions from among $q_1, q_2, \dots q_m$ is said to be odd-formed or even-formed, according as a is odd or even.

If $q_1, q_2, \dots q_m$ be m multenions such that $q_1^2, q_2^2, \dots q_m^2$ are all scalars differing from zero, and each pair is anti-commutative; then, if m is even, the 2^m multiplicative combinations are linearly independent; and, if m is odd, they are independent unless the product $q_1q_2\dots q_m$ of all of them is a scalar, and, if it is a scalar, 2^{m-1} and only 2^{m-1} of the combinations are independent, and these 2^{m-1} independent combinations may be taken as the odd-formed combinations, or instead as the even-formed combinations, or instead as the combinations from which one assigned multenion, such as q_m, is absent. The theorem is true of linities as well as of multenions.

Thus, if n is even, a general multenion cannot be expressed by fewer than 2^n independent scalars. In the case of $n = 4$, it cannot be expressed by fewer than sixteen independent scalars. In the case of $n = 3$, it *can* be expressed by four independent scalars, the same number as in the case of $n = 2$. [But to express it thus, we have to impose the condition that $i_1i_2i_3$ is a scalar such as -1. This is practically Hamilton's procedure in Quaternions.] Either of these cases might be regarded as the case of

Quaternions. I prefer to view Quaternions from another side, namely, as forming the subalgebra of even homogeneities for the case $n = 3$.

In general multenion theory, it is convenient deliberately to prescribe for the case n odd that the 2^n primitive vectoriums shall be independent. This is equivalent to saying that the product $\dot{\iota} = \iota_1\iota_2 \ldots \iota_n$ of all the primitive vectors is *defined* as independent of the scalar 1, even when n is odd.

Whatever be the value of n, multenions with even homogeneities only form a subalgebra with 2^{n-1} units. When n is odd, this subalgebra can always be identified with the next lower multenion system.

When n is even, the number of independent scalars, 2^n, is a perfect square. In this case the algebra is the equivalent of an algebra of linities with the same number. The characteristic equation, therefore, satisfied by a multenion, may be found. When n is even, the degree of the characteristic equation is $2^{\frac{1}{2}n}$. When n is odd, the degree is $2^{\frac{1}{2}(n+1)}$.

In the case of $n = 4$ or 3, the characteristic function may be written in a variety of allied forms, of which the two most important are

$$chf_q(x) \equiv (q-x)\,\mathrm{K}\,(q-x)\,.\,(2\mathrm{V}_0-1)\,[(q-x)\,\mathrm{K}\,(q-x)] \tag{1}$$

$$\equiv (q-x)\,\mathrm{Q}\,(q-x)\,.\,(2\mathrm{V}_0-1)\,[(q-x)\,\mathrm{Q}\,(q-x)]. \tag{2}$$

The coefficient of every power of x is a scalar which it is not difficult to write down in full. Suppose these scalars are m, m', m'', m''', thus

$$chf_q(x) = x^4 - m'''x^3 + m''x^2 - m'x + m. \tag{3}$$

This defines the scalars m. We now have the identity

$$q^4 - m'''q^3 + m''q^2 - m'q + m \equiv 0. \tag{4}$$

In the case of the semi-real system suitable to relativity, $chf_q(0)$ is positive or zero, never negative. The positive value of its fourth root is the natural generalisation of the quaternion tensor $+\sqrt{(q\mathrm{K}q)}$. I have been uncertain whether to denote this positive scalar by $\mathrm{T}q$, thus

$$\left.\begin{aligned} \mathrm{T}q &= +\{q\mathrm{K}q\,.\,(2\mathrm{V}_0-1)\,.\,(q\mathrm{K}q)\}^{\frac{1}{4}} \\ &= +\{q\mathrm{Q}q\,.\,(2\mathrm{V}_0-1)\,.\,(q\mathrm{Q}q)\}^{\frac{1}{4}} \end{aligned}\right\}. \tag{5}$$

An objection to doing so is that this is not Hamilton's meaning in the case of his biquaternion $q+\mathrm{J}r$. His tensor is then imaginary, namely, one of the values of $[(q+\mathrm{J}r)\,\mathrm{K}\,(q+\mathrm{J}r)]^{\frac{1}{2}}$. In our form below, this amounts to saying that the tensor consists of a V_0 part + a V_4 part.

It suffices for our purpose to consider that q only has a tensor when $q\mathrm{K}q$ is a scalar (V_0). For our semi-real relativity case this must be a real scalar,

and in that case Hamilton's meaning, and the meaning given by (5), coalesce. [If $q\mathrm{K}q$ is a scalar, then

$$q\mathrm{K}q = (2\mathrm{V}_0 - 1) \,.\, q\mathrm{K}q,$$

and therefore $\mathrm{T}q$ above $= \{(q\mathrm{K}q)^2\}^{\frac{1}{4}}$.]

The general value of q for which $q\mathrm{K}q$ is a scalar is

$$q = xe^p \text{ where } (1+\mathrm{K})p = 0. \tag{6}$$

The following theorems will now be easily proved by any reader slightly familiar with quaternions.

$$q = \Sigma\iota \mathrm{V}_0\iota^{-1}q = \Sigma\iota^{-1}\mathrm{V}_0\iota q, \tag{7}$$

$$\rho = \sum_{c=1}^{n} i_c\mathrm{V}_0 i_c^{-1}\rho = \sum_{c=1}^{n} i_c^{-1}\mathrm{V}_0 i_c\rho, \tag{8}$$

$$\mathrm{V}_0(q\mathrm{V}_c r) = \mathrm{V}_0(r\mathrm{V}_c q) = \mathrm{V}_0(\mathrm{V}_c q\mathrm{V}_c r), \tag{9}$$

$$\mathrm{V}_0 qr = \mathrm{V}_0 rq, \tag{10}$$

whence the cyclic form

$$\mathrm{V}_0 q_1 q_2 \ldots q_{m-1} q_m = \mathrm{V}_0 q_m q_1 q_2 \ldots q_{m-1},$$

$$\mathrm{V}_0(q\mathrm{R}r) = \mathrm{V}_0(r\mathrm{R}q), \tag{11}$$

$$\mathrm{V}_0(q\mathrm{KR}r) = \mathrm{V}_0(r\mathrm{KR}q). \tag{12}$$

Here R is any capital retroscriptive and KR any capital proscriptive. Thus, for the capital linities, not only are they all commutative, but they are all self-conjugate. Conjugacy is defined and tested as in quaternions. Formally, if ϕ is a given multenion linity, its conjugate, ϕ', is the unique linity for which

$$\mathrm{V}_0 q\phi r = \mathrm{V}_0 r\phi' q, \tag{13}$$

q and r being arbitrary. When $\phi = \phi'$, ϕ is self-conjugate. More generally, for any ϕ, $\frac{1}{2}(\phi+\phi')$ is the self-conjugate part and $\frac{1}{2}(\phi-\phi')$ is the skew part. More general kinds of conjugate present themselves occasionally. For instance, let R be any one of the capital retroscriptives. Q_R, the R-conjugate of ϕ, may be defined by $\mathrm{V}_0 q\mathrm{R}\phi r = \mathrm{V}_0 r\mathrm{R}\phi_\mathrm{R} q$. With these meanings of the conjugate, ϕ', and the R-conjugate, ϕ_R of ϕ, we have

$$\left.\begin{array}{lll} (\phi+\psi)' = \phi'+\psi', & (\phi\psi)' = \psi'\phi', & (\phi')' = \phi \\ (\phi+\psi)_\mathrm{R} = \phi_\mathrm{R}+\psi_\mathrm{R}, & (\phi\psi)_\mathrm{R} = \psi_\mathrm{R}\phi_\mathrm{R}, & (\phi_\mathrm{R})_\mathrm{R} = \phi \\ \phi_\mathrm{R} = \mathrm{R}\phi'\mathrm{R}, & \phi' = \mathrm{R}\phi_\mathrm{R}\mathrm{R} & \end{array}\right\}. \tag{14}$$

The general meaning of a conjugate, ϕ_c, of ϕ may be taken to be that it is a unipotent retroscriptive linity of the linity ϕ. Thus ϕ' is a linity function of the linity ϕ: (1) unipotent because $(\phi')' = \phi$; (2) retroscriptive because $(\phi\psi)' = \psi'\phi'$; (3) linear because $(\phi+\psi)' = \phi'+\psi'$. I have before

me in MS. a full analysis of the properties of such a conjugate, but must not give it here. [Rq stands to q as ϕ' stands to ϕ.] The capital linities are self-R-conjugate as well as self-conjugate. If $\phi q = pqr$, then

$$\phi' q = rqp, \qquad \phi_{R} q = Rp \,.\, q \,.\, Rr, \tag{15}$$

and similarly if $\phi q = p_1 q r_1 + p_2 q r_2 + \ldots$.

Frequently, as in proving the above, we require

$$\text{if} \qquad V_0 pr = V_0 qr, \text{ then } p = q, \tag{16}$$

r being arbitrary, and again

$$\text{if} \qquad V_0 \alpha\rho = V_0 \beta\rho, \text{ then } \alpha = \beta, \tag{17}$$

ρ being arbitrary, and similarly for a ${}_nV_a$ instead of a general multenion or a vector.

When in a semi-real system R is chosen so that $R\iota = \iota^{-1}$ for every ι, we have that if $q = \Sigma x\iota$ and $q' = \Sigma x'\iota$, then

$$V_0 q R q' = \Sigma x x',$$

and in particular

$$V_0 q R q = \Sigma x^2.$$

§ 3. *The Commutation Theorem.*—If ι_a is of homogeneity a, and ι_b of homogeneity b, and if there are exactly c primitive vectors common to the a vectors of ι_a and the b vectors of ι_b it is easy to see that $\iota_a \iota_b$ is of homogeneity $a+b-2c$, and that $\iota_a\iota_b = (-)^{ab-c}\iota_b\iota_a$. Hence

$$V_{a+b-2x} uv = (-)^{ab-x} V_{a+b-2x} vu. \tag{1}$$

[Obviously true when $u = \iota_a$, $v = \iota_b$ as we see by considering separately the two cases $x = c$, $x \neq c$. It is therefore true for u and v. This is a case of proof by reduction to primitive units.]

(1) may be called a first form of the commutation theorem as it shows us a first effect of commuting u and v in uv. (1) at once leads to what may be called the full form

$$\left.\begin{aligned} uv &= (V_{a+b} + V_{a+b-2} + \ldots + V_{a-b})\, uv \\ &= (-)^{ab} (V_{a+b} - V_{a+b-2} + \ldots + [-]^b V_{a-b})\, vu \\ &= (-)^{ab+b} (V_{a-b} - V_{a-b+2} + \ldots + [-]^b V_{a+b})\, vu \end{aligned}\right\}. \tag{2}$$

The following are easy deductions

$$\left.\begin{aligned} \tfrac{1}{2}[uv + (-)^{ab} vu] &= (V_{a+b} + V_{a+b-4} + V_{a+b-8} + \ldots)\, uv \\ \tfrac{1}{2}[uv - (-)^{ab} vu] &= (V_{a+b-2} + V_{a+b-6} + \ldots)\, uv \end{aligned}\right\}, \tag{3}$$

and these may be written in the reverse order with $(-)^{ab+b}$ in place of $(-)^{ab}$. From (3)

$$\left.\begin{aligned} \tfrac{1}{2}(u\omega + \omega u) &= (V_{a+2} + V_{a-2})\, u\omega = (V_{a+2} + V_{a-2})\, \omega u \\ \tfrac{1}{2}(u\omega - \omega u) &= V_a \,.\, u\omega = -V_a \,.\, \omega u \end{aligned}\right\}, \tag{4}$$

which has important applications to infinitesimal rotations. Other special cases are

$$u^2 = (\mathrm{V}_0+\mathrm{V}_4+\mathrm{V}_8+\ldots)\,u^2, \tag{5}$$

$$\omega^2 = (\mathrm{V}_0+\mathrm{V}_4)\,\omega^2, \tag{6}$$

$$\alpha\beta = (\mathrm{V}_0+\mathrm{V}_2)\,\alpha\beta = (\mathrm{V}_0-\mathrm{V}_2)\,\beta\alpha, \tag{7}$$

$$\mathrm{V}_2\alpha\beta = -\mathrm{V}_2\beta\alpha, \qquad \mathrm{V}_0\alpha\beta = \mathrm{V}_0\beta\alpha. \tag{8}$$

$$\mathrm{V}_2\alpha\beta = \tfrac{1}{2}(\alpha\beta-\beta\alpha), \qquad \mathrm{V}_0\alpha\beta = \tfrac{1}{2}(\alpha\beta+\beta\alpha). \tag{9}$$

§ 4. *Rotations of Three Kinds.*—Suppose we find n multenions $p_1, p_2 \ldots p_n$ such that for all purposes of Laws I to IV they may take the place of $i_1, i_2 \ldots i_n$, are we to call the substitution of the multenions p for the vectors i a rotation? In Euclidean space of three dimensions we should not always so affirm. If i, j, k should represent, as sometimes in quaternions they do, three directions in a rigid body we could not move the body so that i became $-i$ and j and k remained unaltered; but they would continue to satisfy the conditions of the laws. We should call the transformation a perversion or reflection rather than a rotation. As the algebra of multenions is (n even) a full linity algebra, or (n odd) a sub-algebra of such a linity algebra, we will assume that given q, q^{-1} can be obtained uniquely in general so that $qq^{-1} = 1 = q^{-1}q$ (though in certain singular cases we have to say instead that q_0 can be found so that $qq_0 = 0 = q_0q$). In the general case we will say that q is vertible (*i.e.*, it has an inverse q^{-1}); in the singular case that q is non-vertible.

Theorem concerning *multenion* rotation. (1) *When q is a given vertible multenion, then in the four fundamental laws the primitive vectors $i_1, i_2, \ldots i_n$ may be replaced respectively by*

$$p_1 = qi_1q^{-1}, \ldots, \qquad p_n = qi_nq^{-1}, \tag{1}$$

and the effect of the replacement is to change any multenion r to

$$r' = qrq^{-1}. \tag{2}$$

(2) *Conversely if $p_1, p_2 \ldots p_n$ are n multenions such that*

$$-1 = p_1^2 = p_2^2 = \ldots = p_n^2, \tag{3}$$

and every pair is anti-commutative, that is

$$p_1p_2+p_2p_1 = 0, \text{ etc.}, \tag{4}$$

and when n is odd

$$p_1p_2\ldots p_n = i_1i_2\ldots i_n, \tag{5}$$

then there exists a vertible multenion, q, for which (1) *and* (2) *are true; r' now meaning the multenion obtained from the given multenion r by replacing $i_1, i_2, \ldots i_n$ by $p_1, p_2, \ldots p_n$ respectively.*

This kind of rotation is scarcely what we seek. For instance, it does not in general leave V_c unchanged, that is, it is not in general true that

$$qV_c r \,.\, q^{-1} = V_c(qrq^{-1}). \tag{6}$$

Multenion rotation qrq^{-1} is operation by a multenion linity $\phi = q(\,)q^{-1}$ such that $\phi'\phi = 1 = \phi\phi'$, and conversely.

Next, imposing the condition that (6) is to be satisfied, we find that q must be the product of any number of vectors, and that if it is such a product (6) is satisfied. We may next find what is the effect of infinitesimally changing the vectors and attempt to build up the finite continuous group of which that infinitesimal transformation is the element. The infinitesimal rotation is the transformation effected by changing q to $q+\frac{1}{2}(\omega q - q\omega)$, where ω is infinitesimal and the corresponding finite rotation is $e^{\omega}(\,)e^{-\omega}$, where ω is finite ω is a perfectly arbitrary ${}_nV_2$, infinitesimal of course for the first case. When ω is not singular ($\omega^2 = 0$) its general form may be put

$$\omega = d\epsilon_1\epsilon_2 + b\epsilon_3\epsilon_4 + \ldots,$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \ldots$ are orthogonal unit vectors

$$e^{\omega} = (\cos a + \epsilon_1\epsilon_2 \sin a)(\cos b + \epsilon_3\epsilon_4 \sin b)\ldots$$

and the factors here are commutative. Putting the last in the form

$$e^{\omega} = \epsilon_1(\epsilon_1^{-1}\cos a + \epsilon_2 \sin a)\,\epsilon_3(\epsilon_3^{-1}\cos b + \epsilon_4 \sin b)\ldots,$$

we see that e^{ω} is the product of an *even* number of vectors. The general form of ω when it is singular is not simple, but I have a complete solution before me in MS.

To show that the infinitesimal rotation is as stated will not occupy too much space. Suppose

$$r' = \alpha\beta\ldots\lambda \,.\, r \,.\, \lambda^{-1}\ldots\beta^{-1}\alpha^{-1} = prp^{-1},$$

where the vectors $\alpha, \beta, \gamma, \ldots$ are assumed to be unit vectors, as that does not affect the transformation. Let $\alpha, \beta, \ldots$ change to $\alpha + d\alpha$, $\beta + d\beta$, $\ldots$, still remaining unit vectors.

$$dr' = d(prp^{-1}) = dpp^{-1} \,.\, r' - r'dpp^{-1}.$$

That dpp^{-1} is a ${}_nV_2$ is straightforwardly proved thus

$$dpp^{-1} = d\alpha\alpha^{-1} + \alpha \,.\, d\beta\beta^{-1} \,.\, \alpha^{-1} + \alpha\beta \,.\, d\gamma\gamma^{-1} \,.\, \beta^{-1}\alpha^{-1} + \ldots.$$

Every term here is a ${}_nV_2$; for example, consider the third term. Since γ and $\gamma + d\gamma$ are unit vectors, $V_0 d\gamma\gamma^{-1} = 0$, and therefore $d\gamma\gamma^{-1}$ is a ${}_nV_2$. It follows that the third term is a ${}_nV_2$. If we put u, u' for r, r' and $\frac{1}{2}d\omega$ for dpp^{-1},

$$du' = \tfrac{1}{2}(d\omega u' - u'd\omega) = V_0 d\omega u'.$$

This is the very simple, very useful form in which true infinitesimal rotation

presents itself in our present subject. A rotation of vectors ρ is a linity $\phi\rho$ of ρ such that $\phi'\phi = 1$.

When without qualification we speak of rotation we shall understand it to mean $e^{\omega}(\;)e^{-\omega}$ (a continuous group).

The skew part of any vector linity, ϕ, is of the form of an infinitesimal rotation. Thus, introduce the ζ notation of quaternions,* by which

$$\mathrm{F}(\zeta, \zeta) = \sum_{c=1}^{n} \mathrm{F}(i_c, i_c), \tag{7}$$

where F is bilinear in its two members. We have

$$\rho = -\zeta \mathrm{V}_0 \zeta \rho, \tag{8}$$

$$\phi'\rho = -\zeta \mathrm{V}_0 \zeta \phi' \rho = -\zeta \mathrm{V}_0 \rho \phi \zeta. \tag{9}$$

The skew part is

$$\tfrac{1}{2}(\phi - \phi')\rho = \tfrac{1}{2}\mathrm{V}_1(\mathrm{V}_2 \zeta \phi \zeta)\rho = \mathrm{V}_1 \omega \rho. \tag{10}$$

Here we have assumed that

$$\mathrm{V}_1(\mathrm{V}_2 \alpha\beta)\gamma = \alpha \mathrm{V}_0 \beta\gamma - \beta \mathrm{V}_0 \gamma\alpha,$$

much as in quaternions. This is not difficult to prove, and we shall in § 7 enunciate much more general theorems of which this is a particular case. The pure part does not similarly reduce. We have

$$\tfrac{1}{2}(\phi + \phi')\rho = -\tfrac{1}{2}(\zeta \mathrm{V}_0 \rho \phi \zeta + \phi \zeta \mathrm{V}_0 \rho \zeta). \tag{11}$$

§ 5. *Vectoriums.* Let

$$\alpha^{(a)} = \alpha_1 \alpha_2 \ldots \alpha_a, \quad \beta^{(b)} = \beta_1 \beta_2 \ldots \beta_b, \quad \gamma^{(c)} = \gamma_1 \gamma_2 \ldots \gamma_c.$$

Then in the expression $\mathrm{V}_{a+b+c}\,(\alpha^{(a)}\beta^{(b)}\gamma^{(c)}$ it is obvious by reduction to primitive units that within the brackets we may at will write either $\alpha^{(a)}$ or $\mathrm{V}_a \alpha^{(a)}$, and $\beta^{(b)}$ or $\mathrm{V}_b \beta^{(b)}$, and $\gamma^{(c)}$ or $\mathrm{V}_c \gamma^{(c)}$; the value of the expression is unaltered by such insertion or removal of V_a, V_b, or V_c. The statement, of course, is in general only true of this particular part, V_{a+b+c} of the product of highest homogeneity. Next, it is similarly evident that if we change, say, α_2 to $x\alpha_2 + y\alpha_2'$ we simply alter the corresponding vectorium similarly, that is

$$\mathrm{V}_a \alpha_1 (x\alpha_2 + y\alpha_2')\alpha_3 \ldots \alpha_a = x \mathrm{V}_a \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_a + y \mathrm{V}_a \alpha_1 \alpha_2' \alpha_3 \ldots \alpha_a.$$

Again, if in the vectorium we interchange two consecutive members such as α_2, α_3 we merely change the sign of the whole. For we may successively write instead of $\alpha_2\alpha_3$

$$\alpha_2\alpha_3, \quad \mathrm{V}_2\alpha_2\alpha_3, \quad -\mathrm{V}_2\alpha_3\alpha_2, \quad -\alpha_3\alpha_2.$$

Finally, by a familiar simple process we may interchange any two members by successive interchanges of neighbours, and as a final result, again, the vectorium changes in sign merely. Thus $\mathrm{V}_a \alpha^{(a)}$ is "combinatorial." The

* See 'Utility of Quaternions.'

common simple properties of a combinatorial multilinear function are probably well known to my readers and may be assumed. [Grassmann apparently first, in a perfectly general manner, appreciated the value of, and fully established these properties. Indeed, $V_a\alpha^{(a)}$ is both algebraically and geometrically identical in meaning with Grassmann's combinatorial "product." I should like for a second time to express my unbounded admiration for Grassmann's great pioneer work, though I cannot help believing that the actual form of the matters he dealt with has been put out of court by later developments.]

We may note the following theorem in passing. It is our form of one of Grassmann's products. If $\acute{\rho}$ is a vectorium of homogeneity, g, then

$$V_{a+b+c}(V_{g+a+b}\acute{\rho}uv)(V_{g+c}w\acute{\rho}) = V_{a+b+c}(V_{g+a}\acute{\rho}u)(V_{g+b+c}vw\acute{\rho}).$$

Of u, v, w the middle one, v, may be passed over from the first to the second bracket-pair.

The following is wanted immediately and is constantly in request. It may be thought of as a passage into or out of the operation of a $V_a(\ )$ [whether as pre-factor or post-factor] of either $\acute{v}$ or p_n. We have

$$\acute{v}V_aq = V_{n-a}\acute{v}q, \qquad V_aq\ .\ \acute{v} = V_{n-a}(q\acute{v}).$$

Also, if p_n is any multenion of homogeneity, n, $p_n = x\acute{v}$, and, therefore, in the present process, p_n behaves exactly as does $\acute{v}$, that is

$$p_nV_aq = V_{n-a}p_nq, \qquad V_aq\ .\ p_n = V_{n-a}\ .\ qp_n.$$

Suppose there are n given linearly independent vectors $\alpha_1, \alpha_2, \dots \alpha_n$. Let $\alpha^{(a)}$ be the product of any a of them, and $\alpha^{(n-a)}$ the product of the rest in such sequence that $V_n\alpha^{(a)}\alpha^{(n-a)}$ always retains the same value. Let

$$\left.\begin{aligned}\acute{\alpha} &= V_a\alpha^{(a)}\\ \mathring{\alpha} &= V_{n-a}\alpha^{(n-a)}\ .\ V_n{}^{-1}\alpha^{(a)}\alpha^{(n-a)}\end{aligned}\right\}. \tag{1}$$

[By a convenient, strictly speaking indefensible, custom, since $V_n{}^{-1}$ really $= \infty$, we mean by $V_n{}^{-1}\alpha^{(a)}\alpha^{(n-a)}$

$$(V_n\alpha^{(a)}\alpha^{(n-a)})^{-1}].$$

There are 2^n linearly independent [proof of linear independence here omitted] values of $\acute{\alpha}$, and they will be called the 2^n vectoriums *formed from* $\alpha_1, \alpha_2, \dots \alpha_n$. [The term may, of course, be used when $\alpha_1, \alpha_2, \dots \alpha_n$ are not linearly independent.] The 2^n corresponding values of $\mathring{\alpha}$, which are also linearly independent, will be called the normal reciprocals of the vectoriums, $\acute{\alpha}$. They are so called because

$$V_0\acute{\alpha}\mathring{\alpha} = 1, \qquad V_0\acute{\alpha}_1\mathring{\alpha} = 0,$$

$\acute{\alpha}_1$ being a second (not corresponding to $\mathring{\alpha}$) $\acute{\alpha}$. [Thus, when $n = 4$, there are

sixteen equations of the type $V_0\acute{a}\grave{a} = 1$ and 240 equations of the type $V_0\acute{a}_1\grave{a} = 0$.] Both equations are proved quite simply from (1) by passing $p_n = V_n^{-1}\alpha^{(a)}\alpha^{(n-a)}$ out of the operation of V_0 in $V_0\acute{a}\grave{a}$.

Let now $q = \Sigma x\acute{a} = \Sigma y\grave{a}$. From the normal reciprocal relations we at once get $x = V_0 q\grave{a}$, $y = V_0 q\acute{a}$. Hence

$$\left.\begin{aligned} q &= \Sigma\acute{a}V_0 q\grave{a} = \Sigma V_a\alpha^{(a)} \,.\, V_n q\alpha^{(n-a)} \,.\, V_n^{-1}\alpha^{(a)}\alpha^{(n-a)} \\ &= \Sigma\grave{a}V_0 q\acute{a} = \Sigma V_0(qV_a\alpha^{(a)}) \,.\, V_{n-a}\alpha^{(n-a)} \,.\, V_n^{-1}\alpha^{(a)}\alpha^{(n-a)} \end{aligned}\right\}, \tag{2}$$

the last result in the first line coming from a new application of the p_n theorem. If in (2) we put $q = u$, getting a more particular result, we should at once see that (2) is analogous to the quaternion theorems

$$\rho S\alpha\beta\gamma = \alpha S\beta\gamma\rho + \ldots = V\beta\gamma S\alpha\rho + \ldots.$$

A particular case $(a = 1)$ of (1) is

$$\left.\begin{aligned} \grave{a}_1 &= V_{n-1}\alpha_2\alpha_3\ldots\alpha_n \,.\, V_n^{-1}\alpha_1\alpha_2\ldots\alpha_3 \\ &= -V_{n-1}\alpha_1\alpha_3\ldots\alpha_n \,.\, V_n^{-1}\alpha_1\alpha_2\ldots\alpha_3 = \text{etc.} \end{aligned}\right\}. \tag{3}$$

This is clearly the solution of the problem. Given

$$S\alpha_c\grave{a}_c = 1, \qquad c = 1, 2, \ldots n,$$
$$S\alpha_b\grave{a}_c = 0, \qquad b \neq c;\ b, c = 1, 2, \ldots n,$$

what are the vectors $\grave{a}_c$ in terms of the vectors α_c? And, of course, we have a precisely similar solution for α_c in terms of $\grave{a}_c$. It appears, then, that the relations between the quantities $\acute{a}$ and the quantities $\grave{a}$ are symmetrical. No doubt we have here established this if only it be evident what is the precise significance of the word "symmetrical" in this connection. The need of the caution may be illustrated thus. If we put $\acute{a} = V_a\alpha_1\alpha_2 \ldots \alpha_n$, we are tempted to say that $\grave{a}$ must be $V_a\grave{a}_1\grave{a}_2 \ldots \grave{a}_a$, but we shall find that we do not thereby always obtain $S\acute{a}\grave{a} = 1$. The true value of $\grave{a}$ is

$$\left.\begin{aligned} \grave{a} &= V_a\grave{a}_a\grave{a}_{a-1}\ldots\grave{a}_1 = QV_a\grave{a}_1\grave{a}_2\ldots\grave{a}_a \\ \text{when} \qquad \acute{a} &= V_a\alpha_1\alpha_2\ldots\alpha_a \end{aligned}\right\}. \tag{4}$$

If the α_c are taken as the i_c, it is obvious that the $\acute{a}$ are the primitive vectoriums, and, from the normal reciprocal relations, it is then obvious that the $\grave{a}$ are also the primitive vectoriums (with different signs in accordance with)

$$\grave{a} = K\acute{a}.$$

Return to the ζ pairs of (7) of § 4. Let $F(q, r)$ be bilinear in two multenions. From the definition of ζ, it is easily seen that

$$F(V_a\zeta_1\zeta_2\ldots\zeta_a, V_a\zeta_1\zeta_2\ldots\zeta_a) = a!\ \Sigma F(\iota_a, \iota_a) = (-)^{\frac{1}{2}a(a+1)}a!\ \Sigma F(\iota_a, \iota_a^{-1}).$$

Abbreviate thus

$$\zeta_1\zeta_2\ldots\zeta_a = \zeta^{(a)}, \quad (-)^{\frac{1}{2}a(a+1)}/a! = a'. \tag{5}$$

[Thus $0' = 1$, $1' = -1$, $2' = -1/2\,!$, $3' = +1/3\,!$ etc., the signs running $+--++--++--....$]

Thus $$\Sigma F(\acute{\iota}_a, \acute{\iota}_a^{-1}) = a' F(V_a\zeta^{(a)}, V_a\zeta^{(a)}). \tag{6}$$

From (6) $$u = \Sigma\acute{\iota}_a V_0 u \acute{\iota}_a^{-1} = a' V_a\zeta^{(a)} \,.\, V_0 u \zeta^{(a)}. \tag{7}$$

In (6) put the second $V_a\zeta^{(a)} = \Sigma\acute{\alpha} V_0 \mathring{\alpha}\zeta^{(a)}$. Thus,

$$a' F(V_a\zeta^{(a)}, V_a\zeta^{(a)}) = a'\Sigma(V_a\zeta^{(a)} V_0 \mathring{\alpha}\zeta^{(a)}, \acute{\alpha}) = \Sigma F(\mathring{\alpha}, \acute{\alpha}).$$

We might have modified the first $V_a\zeta^{(a)}$ instead of the second. Thus collecting results

$$\left.\begin{aligned} a' F(V_a\zeta^{(a)}, V_a\zeta^{(a)}) &= \Sigma F(\acute{\iota}_a, \acute{\iota}_a^{-1}) \\ &= \Sigma F(\mathring{\alpha}, \acute{\alpha}) = \Sigma F(\acute{\alpha}, \mathring{\alpha}) \end{aligned}\right\} \tag{8}$$

($_nC_a$ terms in each summation.)

Just as in quaternions $F(\zeta, \phi\zeta) = F(\phi'\zeta, \zeta)$ so now if ϕ is a $_nV_a$ linity

$$a' F(V_a\zeta^{(a)}, \phi V_a\zeta^{(a)}) = a' F(\phi' V_a\zeta^{(a)}, V_a\zeta^{(a)}). \tag{9}$$

If ϕ is a multenion linity

$$\sum_{a=0}^{n} a' F(V_a\zeta^{(a)}, \phi V_a\zeta^{(a)}) = \sum_{a=0}^{n} a' F(\phi' V_a\zeta^{(a)}, V_a\zeta^{(a)}). \tag{10}$$

For $a = 0$ we must put $a' = 1$, and for each $V_a\zeta^{(a)}$ we must also put 1 [see (7) above].

§ 6. *Extended Vector Linities.*—The theorems of §§ 6, 7 find continuous applications in the multenion treatment of differential invariants. Such treatment may be considered to begin in § 8, and is explicitly introduced in § 9.

Let ϕ be a given vector linity. The extended linity (a multenion linity) which may for most purposes, but not for all, also be denoted by ϕ, is here defined thus. Let $\acute{\iota}_a = i_g i_h \ldots i_l$ be any primitive unit of homogeneity, a, a itself being arbitrary. Then

$$\phi V_a \acute{\iota}_a = V_a \phi i_g \phi i_h \ldots \phi i_l.$$

Now change the n primitive vectors $i_1, i_2, \ldots i_n$ to any other n vectors by such steps as replacing i_1 by $xi_1 + yi_2$ (or, if i_1, i_2 have already changed to ϵ_1, ϵ_2, the change should be from ϵ_1 to $x\epsilon_1 + y\epsilon_2$). Such a change simply multiplies each side of the defining equation (when i_1 occurs in it) by x; the meaning is therefore unaltered. It follows that $\phi V_a \alpha^{(a)} = V_a(\phi\alpha)^{(a)}$ where a is arbitrary and the vectors α are arbitrary. If, then, (ϕ), (ψ) are the extended linities of ϕ, ψ, we see that $(\phi)(\psi)$ is the extended linity of $\phi\psi$, for

$$(\phi)(\psi) V_a i_g i_h \ldots i_l = (\phi) V_a \psi i_g \psi i_h \ldots \psi i_l = V_a \phi\psi i_g \phi\psi i_h \ldots \phi\psi i_l.$$

From this again it follows that the extended linity of ϕ^{-1} is $(\phi)^{-1}$.

Let us use either (ϕ) or ϕ for the extended linity of ϕ for the present. What we have just proved may be put

$$(\phi)(\psi) = (\phi\psi),$$
$$(\phi)^{-1} = (\phi^{-1}).$$

We are now about to prove a similar proposition concerning ϕ' the conjugate, namely

$$(\phi)' = (\phi').$$

By (7) of § 5 we have

$$\phi u = a'\phi \mathrm{V}_a \zeta^{(a)} \mathrm{V}_0 u \zeta^{(a)},$$

or

$$(\phi) u = a'\mathrm{V}_a (\phi\zeta)^{(a)} \mathrm{V}_0 u \zeta^{(a)}. \tag{1}$$

Let u, u_0 be two arbitrary multenions of homogeneity a. Operate on (1) by $\mathrm{V}_0 u_0$ ( )

$$\mathrm{V}_0 u_0 (\phi) u = a'\mathrm{V}_0 u_0 (\phi\zeta)^{(a)} \mathrm{V}_0 u \zeta^{(a)}.$$

Modifying the left by the fundament rule of the meaning of $(\phi)'$, and modify the right by the rule $\mathrm{F}(\zeta, \phi\zeta) = \mathrm{F}(\phi'\zeta, \zeta)$ applicable to a vector linity. We obtain

$$\mathrm{V}_0 u (\phi)' u_0 = a'\mathrm{V}_0 u_0 \zeta^{(a)} \mathrm{V}_0 u (\phi'\zeta)^{(a)}.$$

Since u is arbitrary, by (16) of § 3

$$(\phi)' u_0 = a'\mathrm{V}_a (\phi'\zeta)^{(a)} \mathrm{V}_0 u_0 \zeta^{(a)},$$

which by comparison with (1) proves the assertion that $(\phi)' = (\phi')$.

The following theorems are also true

$$\phi \mathrm{V}_{a+b} uv = \mathrm{V}_{a+b} \phi u \phi v, \tag{2}$$

$$\phi \mathrm{V}_{a-b} uv = \mathrm{V}_{a-b} \phi u \phi'^{-1} v, \tag{3}$$

when ϕ is non-vertible we must change v to $\phi' v$. An alternative, less convenient than (3) generally, is thus

$$\phi \mathrm{V}_{a-b} u \phi' v = \mathrm{V}_{a-b} \phi u v, \tag{4}$$

but this alternative is true without qualification.

$\acute{v}$ having its usual meaning $(i_1, i_2, \ldots i_n)$ it is easy to see that $\phi\acute{v} = m\acute{v}$ where m is the determinant of ϕ. Putting $u = \acute{v}$ in (3) and (4) we obtain

$$\left.\begin{aligned} \acute{v}^{-1}\phi(\acute{v}v) &= m\phi'^{-1}v \\ \text{or} \qquad \acute{v}^{-1}\phi(\acute{v}\phi' v) &= mv \end{aligned}\right\}. \tag{5}$$

[*Added February,* 1921.—Closely connected with the extension in § 6 from vector operation to multenion operation, of a given vector linity, is a second kind of such extension. If ϕ_{E} is the extended (or, as I now prefer to say

extensive) form of ϕ of § 6, its fundamental property, showing its dependence (in a *multiple* manner) on ϕ, is that

$$\phi_{\mathrm{E}}\,\mathrm{V}_a\alpha_1\alpha_2\ldots\alpha_a = \mathrm{V}_a\phi\alpha_1\phi\alpha_2\ldots\phi\alpha_a. \tag{1}$$

Similarly, if ψ_e is the new extensive form of ϕ, its fundamental property showing its dependence (in an *additive* manner) on ϕ is that

$$\psi_e\mathrm{V}_a\alpha_1\alpha_2\ldots\alpha_a = \mathrm{V}_a(\psi\alpha_1)\,\alpha_2\ldots\alpha_a + \mathrm{V}_a\alpha_1(\psi\alpha_2)\,\alpha_3\ldots\alpha_a + \ldots. \tag{2}$$

The dependence is called multiple or additive because

$$(\phi\Phi)_{\mathrm{E}} = \phi_{\mathrm{E}}\Phi_{\mathrm{E}}, \qquad (\psi+\Psi)_e = \psi_e+\Psi_e. \tag{3}$$

We may speak briefly of ϕ_{E} as the extensive of ϕ and ψ_e as the subextensive of ψ (using extensive and subextensive both as nouns and adjectives; compare offensive and defensive; compare also Whitehead's translation of "Ausdehnungslehre" as "Calculus of extension").

For both kinds of extensive it is true, when ϕ_{E} and ψ_e operate on a mere vector, that

$$\phi_{\mathrm{E}}\rho = \phi\rho, \qquad \psi_e\rho = \psi\rho, \tag{4}$$

so that quite frequently it is convenient to use ϕ for ϕ_{E} and on other occasions to use ψ for ψ_e. [We should very rarely with a given ϕ want on a single occasion to use both ϕ_{E} and ϕ_e, but if we do we must have some mark to distinguish them, since they have diverse meanings.]

Although the properties of a subextensive linity may be derived directly from its fundamental property defined above, they may instead be derived from those of an extensive by the methods connecting an infinitesimal with a finite transformation. Thus we may put

$$\left.\begin{aligned} \phi &= 1+\dot{\phi}dt = 1+\psi dt \\ (1+\psi dt)_{\mathrm{E}} = \phi_{\mathrm{E}} &= (1+\dot{\phi}dt)_{\mathrm{E}} = 1+\psi_e dt \end{aligned}\right\}, \tag{5}$$

where dt is an arbitrary infinitesimal scalar. Thus from former results we at once have

$$\mathrm{V}_c\psi_e = \psi_e\mathrm{V}_c = \mathrm{V}_c\psi_e\mathrm{V}_c, \tag{6}$$

$$(\psi_e)' = (\psi')_e, \tag{7}$$

$$\psi_e\mathrm{V}_{a+b}uv = \mathrm{V}_{a+b}(\psi_e u\,.\,v+u\psi_e v), \tag{8}$$

$$\psi_e\mathrm{V}_{a-b}uv = \mathrm{V}_{a-b}(\psi_e u\,.\,v-u\psi_e' v), \tag{9}$$

$$\psi_e\acute{v} = \psi_e'\acute{v} = -\acute{v}\,.\,\mathrm{V}_0\zeta\psi\zeta, \tag{10}$$

$$\psi_e v = -\mathrm{V}_b\psi\zeta\mathrm{V}_{b-1}\zeta v = -\mathrm{V}_b(\mathrm{V}_0\psi\zeta\zeta\,.\,v-\zeta\mathrm{V}_{b+1}\psi\zeta v), \tag{11}$$

$$\acute{v}^{-1}\psi_e(\acute{v}v) = -(\psi_e'+\mathrm{V}_0\zeta\psi\zeta)\,v = -\mathrm{V}_b\psi\zeta\mathrm{V}_{b+1}\zeta v, \tag{12}$$

the last being obtained by putting $u = \acute{v}$ in (9) above.

If ϕ and ψ are given vector linities then $\phi_E \psi_e \phi_E^{-1}$ is a subextensive that is it is the subextensive of $\phi\psi\phi^{-1}$. For by (11)

$$(\phi\psi\phi^{-1})_e v = -V_b \phi\psi\phi^{-1}\zeta V_{b-1}\zeta v = -\phi_E V_b \psi\zeta V_{b-1}\zeta\phi_E^{-1} v,$$

by applications of both the equations [see (2) and (3) of § 6],

$$\phi_E V_{a+c} uw = V_{a+c}\phi_E u \phi_E w, \qquad \phi_E V_{c-a} uw = V_{c-a}\phi_E'^{-1} u \phi_E w.$$

If ϕ is of assigned dimensions, ϕ_e is of the same, but not ϕ_E. $\phi_E u$, in so far as its dimensions depend on ϕ_E and not on u, are a times the dimensions of ϕ.]

§ 7. *Formulæ Analogous to the Quaternion Formulæ for* $V\alpha V\beta\gamma$ *and* $V\alpha\beta\gamma$.—Let $\alpha_1, \alpha_2, \ldots, \alpha_{a+b}$ be $a+b$ given vectors and let $\alpha^{(a)}$ be the product of any a of them and $\alpha^{(b)}$ the product of the rest in such a sequence that $V_{a+b}\alpha^{(a)}\alpha^{(b)}$ always has the same value. Then u being as usual

$$V_b \,.\, u V_{a+b}\alpha^{(a)}\alpha^{(b)} = V_b (V_{a+b}\alpha^{(b)}\alpha^{(a)} \,.\, u) = \Sigma V_b \alpha^{(b)} V_0 \alpha^{(a)} u. \tag{1}$$

Note that $\alpha^{(a)}$ occupies the middle position throughout. If in this we put $b = n-a$ it easily leads to the last form of (2) of § 5.

u and v being as usual and ρ any vector,

$$V_{1+(a\sim b)} \,.\, u V_{b+1}\rho v = V_{1+(a\sim b)}[V_{a-1}u\rho \,.\, v + (-)^a \rho V_{a\sim b} uv], \tag{2}$$

$$V_{1+(a\sim b)}(V_{a+1}u\rho \,.\, v) = V_{1+(a\sim b)}[u V_{b-1}\rho v + (-)^a \rho V_{a\sim b} uv], \tag{3}$$

$$\left.\begin{aligned} V_{1+(a\sim b)} u\rho v &= V_{1+(a\sim b)}[V_{a-1}u\rho \,.\, v + u V_{b-1}\rho v + (-)^a \rho V_{a\sim b} uv] \\ &= V_{1+(a\sim b)}[V_{a+1}u\rho \,.\, v + u V_{b+1}\rho v - (-)^a \rho V_{a\sim b} uv] \end{aligned}\right\}, \tag{4}$$

$$\left.\begin{aligned} V_{a+b-1} u\rho v &= V_{a+b-1}[V_{a-1}u\rho \,.\, v + u V_{b+1}\rho v + (-)^a \rho V_{a+b} uv] \\ &= V_{a+b-1}[V_{a+1}u\rho \,.\, v + u V_{b-1}\rho v - (-)^a \rho V_{a+b} uv] \end{aligned}\right\}. \tag{5}$$

To make practical applications of multenions it is absolutely essential that facility should be gained in using these formulæ. The following comments are to aid in the acquirement of the requisite facility. I do not attempt to work without a list of at least the four equations (4) and (5) before me. Each of those four generates two others. (2) and (3) are those two derived from the first equation (4). If the second term on the right of this equation (call it (4a)) is removed to the left it can be made to merge with the left and equation (2) is produced. Similarly, if the first term on the right of (4a) is removed to the left equation (3) is produced. Now as after a little practice, with (4a) before the eye, the whole of this process can be gone through mentally and the desired modification of an expression under treatment can forthwith be written down, (2) and (3) are strictly speaking not required in the working list. Each of the four equations (4) and (5) is similar in the points described to (4a). The reader who desires to acquire this facility could not do better than begin by writing out all the twelve equations implied.

Attend now only to (4) and (5). If the two equations (4) are added together one reaches a mere identity almost immediately. Therefore, either of the equations (4) readily follows from the other. Exactly similarly either of the equations (5) follows from the other. To obtain one of (4) from one of (5), place $\acute{v}$ as a post factor to one of (5) and then pass it into the various operators $\mathrm{V}x(\ )$ till it reaches v, noting how those operators must be changed in the process. When $\acute{v}$ has come to juxtaposition with v, change $v\acute{v}$ into w so that $n-b=c$ and get rid of b, leaving a and c. The completion of the verification is simple. I have verified that it is possible to prove either of (5) by reduction to primitive units. I have found no simple proof. Actually I established one of the twelve equations by taking ρ to be one of the vectors α_c, say α_{a+b} in (1). Perhaps this is the simplest method.

Because the space I have assigned myself does not permit otherwise, I must now omit several chapters in the MS. treatise from which I have been summarising, but I find one matter requires insertion before I pass to the part of the subject lending itself to applications to differential invariants.

If ϕ is a multenion linity its characteristic function $chf_\phi(x)$ is a certain n-ic in the arbitrary scalar symbol x. Another name for it is that it is the determinant of $x-\phi$, and from the point of view suggested it may be denoted by $|x-\phi|$. In our present notation this is given by the identity

$$\left.\begin{aligned} chf_\phi(x) &\equiv |x-\phi| \\ &\equiv n'\mathrm{V}_n\zeta^{(n)}\,.\,\mathrm{V}_n[(x-\phi)\zeta]^{(n)} \\ &\equiv x^n-m_1x^{n-1}+\ldots+(-)^n m_n \end{aligned}\right\}, \tag{6}$$

[n' is the same function of n as a' is of a in (5) of § 5, that is $n'=(-)^{\frac{1}{2}n(n+1)}/n!$ m_n here and in quaternions is generally written as m]. Several forms derived from (6) are useful in the present subject. They are expressions for m_a of which it is to be remembered that m_n is a specially important case. There is (1) a fundamental form in ζ, (2) a form involving n given linearly independent vectors $\alpha_1, \alpha_2, \ldots \alpha_n$, (3) a form dependent on the general expression for ϕ

$$\phi\epsilon = -\beta_1\mathrm{V}_0\epsilon\alpha_1-\beta_2\mathrm{V}_0\epsilon\alpha_2-\ldots = -\Sigma\beta\mathrm{V}_0\epsilon\alpha, \tag{7}$$

and (4) the particular case of the last for which

$$\phi\epsilon = -\mathrm{V}_0\epsilon\nabla. \tag{8}$$

In the order mentioned, these forms are

$$\left.\begin{aligned} m_a &= a'\mathrm{V}_0(\mathrm{V}_a\zeta^{(a)}\phi\mathrm{V}_a\zeta^{(a)}) \\ &= \Sigma\,.\,\mathrm{V}_n[\mathrm{V}_a(\phi\alpha)^{(a)}\,.\,\mathrm{V}_{n-a}\alpha^{(n-a)}]\,.\,\mathrm{V}_n^{-1}\alpha^{(a)}\alpha^{(n-a)} \\ &= \Sigma\,.\,\mathrm{V}_0\mathrm{V}_a\alpha^{(a)}\mathrm{K}\mathrm{V}_a\beta^{(a)} \\ &= a'\mathrm{V}_0(\mathrm{V}_a\nabla^{(a)}\mathrm{V}_a\epsilon^{(a)}) \end{aligned}\right\}. \tag{9}$$

The last three are all deducible from the first. Another form could at once be written down from the first, involving the vectors α and the vectors $\hat{a}$.

For our applications below the fourth form is important. For this case

$$F(\zeta, \phi\zeta) = F(\nabla_1, \epsilon_1). \tag{10}$$

From this
$$V_0\zeta\phi\zeta = V_0\nabla\epsilon, \qquad V_2\zeta\phi\zeta = V_2\nabla\epsilon, \tag{11}$$

$$|\phi| = m_n = n'V_0 \,.\, V_n\nabla^{(n)}V_n\epsilon^{(n)}. \tag{12}$$

[*Added February*, 1921.—The formulæ analogous to the quaternion formula for $V\alpha V\beta\gamma$ are here greatly generalised and reduced to a form in which they are easy to remember and to apply in particular cases.

In accordance with (3) of § 3 let

$$\left.\begin{aligned} \tfrac{1}{2}[uv+(-)^{ab}vu] &= (V_{a+b}+V_{a+b-4}+V_{a+b-8}+\ldots)\,uv = \Lambda uv \\ \tfrac{1}{2}[uv-(-)^{ab}vu] &= (V_{a+b-2}+V_{a+b-6}+\ldots)\,uv = \Lambda' uv \end{aligned}\right\}. \tag{1}$$

The parts of Λuv, and Λuv itself, may be said to be concordant with V_{a+b}, those of $\Lambda' uv$ concordant with V_{a+b-2}. Similarly, let the parts of uvw, as follows,

$$\left.\begin{aligned} \tfrac{1}{2}(uvw+(-)^{bc+ca+ab}wvu) &= \Lambda uvw \\ \tfrac{1}{2}(uvw-(-)^{bc+ca+ab}wvu) &= \Lambda' uvw \end{aligned}\right\}, \tag{2}$$

be called concordant with V_{a+b+c} and $V_{a+b+c-2}$ respectively. [That Λuvw as defined by (2) is the same as $(V_{a+b+c}+V_{a+b+c-4}+\ldots)\,uvw$ may be proved from the properties of K.

$$K(uvw) = KwKvKu = (-)^{\frac{1}{2}\Sigma a(a+1)}wvu.$$

Hence
$$2\Lambda(uvw) = \{1+(-)^{\frac{1}{2}(a+b+c)(a+b+c+1)}K\}\,uvw].$$

It is important for present purposes to note the various forms of such an expression as

$$(\Lambda \text{ or } \Lambda') \,.\, u(\Lambda \text{ or } \Lambda')\,vw,$$

obtained by mere commutations, and especially what are the commutation factors in the various cases. There are two standard cases from which all others may be readily derived. They are

$$\Lambda vu = (-)^{ab}\Lambda uv, \qquad \Lambda(\Lambda vw \,.\, u) = (-)^{a(b+c)}\Lambda(u\Lambda vw),$$

and we express them by saying that $(-)^{ab}$ is the commutation factor of Λuv, and that $(-)^{a(b+c)}$ is the commutation factor for u and Λvw in $\Lambda(u\Lambda vw)$. The addition of an accent to any Λ clearly multiplies the commutation factor by -1. Thus

$$\Lambda' vu = -(-)^{ab}\Lambda' uv,$$

$$\Lambda(\Lambda' vw \,.\, u) = -(-)^{a(b+c)}\Lambda(u\Lambda' vw),$$

$$\Lambda'(\Lambda' vw \,.\, u) = (-)^{a(b+c)}\Lambda'(u\Lambda' vw), \text{ etc.}$$

Thus, if z, y, are independently equal either to 0 or 1, and if we put $\Lambda^{(x)}$ for Λ or Λ' according as x is 0 or 1, and similarly for $\Lambda^{(y)}$, we have by commutations four forms of $\Lambda^{(x)} . u\Lambda^{(y)}vw$, namely,

$$\begin{aligned}\Lambda^{(x)}(u\Lambda^{(y)}vw) &= (-)^{y+bc}\Lambda^{(x)}(u\Lambda^{(y)}wv)\\ &= (-)^{x+y+a(b+c)}\Lambda^{(x)}(\Lambda^{(y)}vw \,.\, u) = (-)^{x+bc+ca+ab}\Lambda^{(x)}(\Lambda^{(y)}wv \,.\, u).\end{aligned} \tag{3}$$

It is clear that for all present purposes u, v, w may be generalized in meaning. Hitherto they have meant multenions of the forms $V_a p$, $V_b q$, $V_c r$; but they may here be extended to mean multenions concordant with V_a, V_b, V_c, that is

$$\left.\begin{aligned} u &= (V_a+V_{a+4}+V_{a-4}+V_{a+8}+V_{a-8}+\ldots)p,\\ v &= (V_b+V_{b+4}+V_{b-4}+\ldots)q, \quad w = (V_c+V_{c+4}+\ldots)r\end{aligned}\right\}. \tag{4}$$

Let now Vq stand for any $V_x q$, a *single* selected part, which is concordant with Λq; and $V'q$ for any single part which is concordant with $\Lambda' q$. Then the following 8 statements are identities, and each of the 8 has 64 variants obtained by such commutations as we have just been considering. Thus, given u, v, w, there are in all 512 variants, and in each of these V, or V′, has in general a multitude of values. In each of the 8 identities we may add together the individual parts indicated by the V or V′, which means that V or V′ may in every case be replaced by Λ or Λ' respectively. For standard reference forms those written seem preferable. The identities are in meaning (not separately, but taken all together), though not in form, symmetrical in u, v, w. This will appear in the course of the proof given later.

$$\left.\begin{aligned} V(\Lambda uv \,.\, w) &= V(u\Lambda vw+(-)^{ab}v\Lambda' uw)\\ &= V(u\Lambda' vw+(-)^{ab}v\Lambda uw),\\ V(\Lambda' uv \,.\, w) &= V(u\Lambda vw-(-)^{ab}v\Lambda uw)\\ &= V(u\Lambda' vw-(-)^{ab}v\Lambda' uw),\\ V'(\Lambda uv \,.\, w) &= V'(u\Lambda vw+(-)^{ab}v\Lambda uw)\\ &= V'(u\Lambda' vw+(-)^{ab}v\Lambda' uw),\\ V'(\Lambda' uv \,.\, w) &= V'(u\Lambda vw-(-)^{ab}v\Lambda' uw)\\ &= V'(u\Lambda' vw-(-)^{ab}v\Lambda uw)\end{aligned}\right\}. \tag{5}$$

The following remarks will probably render these easy to remember and apply: (1) Counting the accent of a V′ only *once* when it occurs in an equation, the number of accents in each of the eight equations of three terms is odd, that is, it is 1 or 3. (2) In these standard forms the sequence u, v, w is adopted as far as practicable, that is to say, wholly in the term on the left and the first term on the right; whereas in the second term on the right

v is forced away from the middle position, the sequence u, w is still maintained. (3) When these rules of sequence are adopted the signs to be given to each term are specially easy of determination; when terms on opposite sides of the equation have the same sequence, or by commutation are brought to the same sequence, they have or are brought to have the same sign; and when on the same side to have opposite signs. Thus, as written in (5), the term on the left and the first term on the right, having the same sequence, have the same (plus) sign; and again, the term on the left and the second term on the right are brought to the same sequence by commuting u and v on the left. This last determines the sign, $\pm(-)^{ab}$ in the second term on the right.

We will now throw the eight separate cases of (5) into one form, and henceforth replace V and V′ by Λ and Λ'. Let g, x, y, z each independently be equal to 0 or 1. Then all the cases of (5) are included in

$$\left.\begin{array}{ll} & \Lambda^{(g)}(\Lambda^{(z)}uv\,.\,w) = \Lambda^{(g)}(u\Lambda^{(x)}vw + (-)^{z+ab}v\Lambda^{(y)}uw) \\ \text{provided} & g+x+y+z = 1 \text{ or } 3 \end{array}\right\}. \qquad (6)$$

Put x', y', z' for the commutation factors of $\Lambda^{(x)}vw$, $\Lambda^{(y)}wu$, $\Lambda^{(z)}uv$ respectively, that is

$$x' = (-)^{x+bc}, \qquad y' = (-)^{y+ca}, \qquad z' = (-)^{z+ab} \qquad (7)$$

It is easy from (3) to prove that $-y'z'$, $-z'x'$, $-x'y'$ are the commutation factors for u and $\Lambda^{(x)}vw$ in $\Lambda^{(g)}\,.\,u\Lambda^{(x)}vw$; and for the like single symbol and coupled pair in $\Lambda^{(g)}\,.\,v\Lambda^{(y)}wu$ and $\Lambda^{(g)}\,.\,w\Lambda^{(z)}uv$ respectively.

To see that (6) is symmetrical in u, v, w, make the commutations necessary to obtain the cyclic order u, v, w throughout; first with the single symbol in the first place throughout, and secondly in the last place throughout. Also put all the terms on one side of the equation and write the three terms in the order for which in the first term u occupies the middle position, in the second v does so, and in the third w. Notice that the coefficients x', y', z' thus come in both arrangements to have their alphabetic order. Thus

$$\Lambda^{(g)}(x'w\Lambda^{(z)}uv + y'u\Lambda^{(x)}vw + z'v\Lambda^{(y)}wu) = 0, \qquad (8)$$

$$\Lambda^{(g)}(x'\Lambda^{(y)}wu\,.\,v + y'\Lambda^{(z)}uv\,.\,w + z'\Lambda^{(x)}vw\,.\,u) = 0. \qquad (9)$$

To show that these are equivalent to (6), we have only to verify for any two terms in each equation that when brought to the same sequence they have opposite signs, and this is quite easy to do. [Commute w with $\Lambda^{(z)}uv$ in the first term of (8); it comes to agreement in sequence and disagreement in sign with the second term.]

We will now prove (8). In the first term in (8), $-x'y'$ is one commutation factor and z' is the other. Hence four times this term is

$$\begin{aligned} 4x'\Lambda^{(g)}w\Lambda^{(z)}uv &= 2x'(w\Lambda^{(z)}uv - x'y'\Lambda^{(z)}uv \,.\, w) \\ &= 2(x'w\Lambda^{(z)}uv - y'\Lambda^{(z)}uv \,.\, w) \\ &= x' \,.\, wuv + z'x' \,.\, wvu - y'uvw - y'z' \,.\, vuw. \end{aligned}$$

We see by cyclic change from this that four times the second and third terms are

$$y' \,.\, uvw + x'y' \,.\, uwv - z' \,.\, vwu - z'x' \,.\, wvu$$

and

$$z' \,.\, vwu + y'z' \,.\, vuw - x' \,.\, wuv - x'y' \,.\, uwv.$$

The sum of these three expressions is identically zero, as is seen by inspection.

We can reproduce the equations (5) in other notations. Thus, let

$$vu = \mathrm{C}uv, \qquad wvu = \mathrm{C}uvw,$$

$$\tfrac{1}{2}(1+\mathrm{C})(\ ) = \overline{\mathrm{C}}(\ ), \qquad \tfrac{1}{2}(1-\mathrm{C})(\ ) = \overline{\mathrm{C}}'(\ ),$$

$$\tfrac{1}{2}(1+\mathrm{K})q = \overline{\mathrm{K}}q, \qquad \tfrac{1}{2}(1-\mathrm{K})q = \overline{\mathrm{K}}'q,$$

$$\tfrac{1}{2}(1+\mathrm{Q})q = \overline{\mathrm{Q}}q, \qquad \tfrac{1}{2}(1-\mathrm{Q})q = \overline{\mathrm{Q}}'q.$$

Then, similar to (6), the following is true

$$\left.\begin{aligned} &\overline{\mathrm{C}}^{(g)}(\overline{\mathrm{C}}^{(z)}uv \,.\, w) = \overline{\mathrm{C}}^{(g)}(u\overline{\mathrm{C}}^{(z)}vw + (-)^{(z)}v\overline{\mathrm{C}}^{(y)}uw) \\ \text{provided}\quad &g+x+y+z = 1 \text{ or } 3 \end{aligned}\right\}. \tag{10}$$

The application of the rule of signs is rather improved, but in most cases the reverse is decidedly the case in the application of the proviso. In (10) *within the brackets* each C may be replaced by K, or, instead, by Q, but outside the brackets $\overline{\mathrm{C}}^{(g)}$ must be retained; also with K or Q in place of C the index z of $(-)^z$ must be changed to

$$z + \tfrac{1}{2}[a(a \pm 1) + b(b \pm 1)],$$

the upper sign being taken with K and the lower with Q.

(10) is more closely analogous than (6) to the quaternion formula for $\mathrm{V}\alpha\mathrm{V}\beta\gamma$, but we must remember that in quaternions

$$\alpha\beta + \beta\alpha = 2\mathrm{S}\alpha\beta, \qquad \alpha\beta\gamma + \gamma\beta\alpha = 2\mathrm{V}\alpha\beta\gamma,$$

$$\alpha\beta - \beta\alpha = 2\mathrm{V}\alpha\beta, \qquad \alpha\beta\gamma - \gamma\beta\alpha = 2\mathrm{S}\alpha\beta\gamma,$$

so that inside the brackets $\overline{\mathrm{C}}$, $\overline{\mathrm{C}}'$ are analogous to S, V, whereas outside they are analogous to V, S.

Of these different forms (5) is without doubt the best for general use, but the others are occasionally convenient.

Putting $\qquad \Lambda - \Lambda' = \mathrm{L} \qquad$ and $\qquad \mathrm{L}u = u = \mathrm{C}u,$

we have that C, K, Q, are all strictly retroscriptive that is

$$Cuvw = CwCvCu, \qquad Kpqr = KrKqKp, \text{ etc.,}$$

but L is not, though it has the analogous property

$$Luv = (-)^{ab}LvLu, \qquad Luvw = (-)^{bc+ca+ab}LwLvLu.$$

These last properties are easily extended to any number of symbols, such as u, v, w. To pass to higher numbers than three with (5), or say (6), is much more troublesome. Here is an example of passing to four symbols. Let u' v' be multenions concordant with $V_{a'}$, $V_{b'}$. In $\Lambda^{(g)}(\Lambda^{(h)}uv\Lambda^{(h')}v'u')$, [where outside the brackets $\Lambda^{(g)}$ is concordant with $V_{a+b+a'+b'-2g}$] first treat $\Lambda^{(h')}v'u'$ as a single symbol, and secondly treat $\Lambda^{(h)}uv$ in that manner. We obtain

$$\left.\begin{aligned} &\Lambda^{(g)}(\Lambda^{(h)}uv\Lambda^{(h')}v'u' \\ &\quad = \Lambda^{(g)}(u\Lambda^{(x)}\,.\,v\Lambda^{(h')}v'u' + (-)^{h+ab}v\Lambda^{(y)}\,.\,u\Lambda^{(h')}v'u' \\ &\quad = \Lambda^{(g)}(\Lambda^{(x')}\,.\,[\Lambda^{(h)}uv\,.\,v']u' + (-)^{h'+a'b'}\Lambda^{(y)}\,.\,[\Lambda^{(h)}uv\,.\,u']v') \\ &\text{provided both} \qquad g+(h+h')+(z+y) \\ &\text{and} \qquad\qquad g+(h+h')+(x'+y') \text{ are odd} \end{aligned}\right\}. \qquad (11)$$

In the equation of four terms provided by (11) there would be thirty-two cases corresponding to the eight of (5), because double value sare assignable independently to g, h, h' x, x'. In each of the equations of three terms provided by (11) there would be sixteen cases.]

§ 8.—*Integration.*—Let $x_1i_1+x_2i_2+\ldots+x_ni_n = \rho$ stand for an independent variable position vector of a point in Euclidian space of n dimensions. Let a point trace a curve of which an element is $d\rho\,(=d\rho_1)$. Let this curve move and trace a two-dimensional tract. In the process let an element of the path of the element $d\rho\,(=d\rho_1)$ be $d\rho_2$. The element of the two-dimensional region traced hereby is given by $V_2d\rho_1d\rho_2$. Let this be continued till a $(b+1)$-dimensional tract has been traced.

Let us consider a space integral over the complete boundary (a b-tract) of the $(b+1)$-tract and express it as a space integral over the $(b+1)$-tract. To do this, starting from any point of the $(b+1)$-tract, let an elementary closed b-tract expand till it has traversed the whole $(b+1)$-tract by reaching its boundary. The process could be described in much more detail. We suppose the $(b+1)$-tract to be eventually filled with parallelepipedal elements. First, for each such element the connection between integral over boundary (of element) is expressed as a multiple of $V_{b+1}d\rho_1d\rho_2\ldots d\rho_{b+1}$, denoting that element of $(b+1)$-tract. These are summed and in this sum contributions from the common boundary of two elements cancel. The direction indicated by $d\rho_{b+1}$ is to be *into* the region bounded by the element

$$d\dot{\rho}_b = V_bd\rho_1d\rho_2\ldots d\rho_b.$$

With these conventions the following is true

$$\int^{(b)}\int \phi d\mathring{\rho}_b = \int^{(b+1)}\int \phi_g \mathrm{V}_b d\mathring{\rho}_{b+1}\nabla_g, \tag{1}$$

where

$$\nabla = \sum_{c=1}^{n} i_c \mathrm{D}_{x_c}, \tag{2}$$

$\phi d\mathring{\rho}_b$ is the element of boundary-integral due to the element of boundary $d\mathring{\rho}_b$, and $d\mathring{\rho}_{b+1}$ is an element of the $(b+1)$-tract. The form of ϕ is a function of position. The suffix g means that ∇ acts on this function ϕ.

We will now put the integral (1) in a second form. Let $\mathring{\upsilon} = \mathrm{U}d\mathring{\rho}_n$ ($\mathrm{U}d\mathring{\rho}_n\,\mathrm{T}d\mathring{\rho}_n = d\mathring{\rho}_n$, $(\mathrm{T}d\mathring{\rho}_n)^2 = \pm\,(d\mathring{\rho}_n)^2$, $\mathrm{T}d\mathring{\rho}_n$ a positive real scalar). Also let $d\mathring{\rho}_b = \mathring{\upsilon} d\varsigma_{n-b}$. $d\varsigma_0$ is thus a positive scalar, which measures an element of hyper *bulk*. $d\varsigma_1$ is a vector which measures an element of hyper *area*. Write db for $d\varsigma_0$ and $d\alpha$ for $d\varsigma_1$; then

$$\left.\begin{aligned} db &= \mathrm{V}_0 d\alpha\, d\rho_n \\ d\mathring{\rho}_n &= \mathrm{V}_n d\mathring{\rho}_{n+1} d\rho_n \end{aligned}\right\}. \tag{3}$$

follows from

Since db is positive and $d\rho_n$ points inwards, $d\alpha$ points outwards in harmony with usual convention.

Equation (1) in the $d\varsigma$ notation becomes

$$\int^{(n-a)}\int \psi d\varsigma_{n-a} = \int^{(n-a+1)}\int \psi_g \mathrm{V}_a d\varsigma_{a-1}\nabla_g. \tag{4}$$

The extreme cases of (1) and (4) are when $b = 1$ and $a = 1$ that is

$$\int \phi d\rho = \iint \phi_g \mathrm{V}_1 d\mathring{\rho}_2 \nabla_g, \tag{5}$$

$$\int^{(n-1)}\int \psi d\alpha = \int^{(n)}\int \psi_g \nabla_g. \tag{6}$$

In (1) put $\phi d\mathring{\rho}_b =$ a scalar $= \mathrm{V}_0 v d\mathring{\rho}_b$ and make similar changes in the other three. Thus (1) and (4) become

$$\int^{(b)}\int \mathrm{V}_0 v d\mathring{\rho}_b = \int^{(b+1)}\int \mathrm{V}_0 d\mathring{\rho}_{b+1} \mathrm{V}_{b+1}\nabla v, \tag{7}$$

$$\int^{(n-a)}\int \mathrm{V}_0 uk d\varsigma_a = \int^{(n-a+1)}\int \mathrm{V}_0 d\varsigma_{a-1} \mathrm{V}_{a-1}\nabla (ku), \tag{8}$$

uk has been written in place of a mere u for subsequent convenience. k is supposed to be a given scalar function of position of the nature of density. Thus, if by a general strain ρ becomes ρ', $d\rho$ becomes $d\rho'$, and db becomes db', then k becomes k' where

$$kdb = k'db', \tag{9}$$

k is used in the transformation of (6). (5) and (6) become

$$\int \mathrm{V}_0 \sigma d\rho = \iint \mathrm{V}_0 d\mathring{\rho}_2 \mathrm{V}_2 \nabla\sigma, \tag{10}$$

$$\int^{(n-1)}\int \mathrm{V}_0 \tau k d\alpha = \int^{(n)}\int \mathrm{V}_0 \nabla (k\tau)\, db. \tag{11}$$

We are about to apply these theorems to differential invariants. It may surprise some readers that there is so much concerning invariants which is

independent of any conception of differential quadratic forms. I was acquainted with all the theorems here given to the end of § 10 years before I had heard of quadratic differential forms. My first acquaintance with such forms was after the MS. treatise from which I extract them had been completed. That first (and up to this date my only) acquaintance with such forms was through Prof. Wright's delightful Cambridge tract on the subject. How then did they originate? From a study of Maxwell's suggestions concerning intensities and fluxes. Till the last three months I never looked at them from any other point of view. I gave elaborate applications of their three-dimensional form, twenty-seven years ago, to electrical problems.*

§ 9. *Covariants and Contravariants.*—Let ρ' be a vector function of the independent vector ρ. We may think of this as representing a displacement of every point of a medium filling our n-dimensional Euclidean space from the position ρ to the position ρ', or we may think of it in its pure mathematical aspect as a mere mathematical transformation of the coordinates $x_c = -\mathrm{V}_0 i_c \rho$ into the coordinates $x_c' = -\mathrm{V}_0 i_c \rho'$.

Corresponding to an arbitrary increment $d\rho$ of ρ, there is an increment $d\rho'$ of ρ' given by

$$d\rho' = -\mathrm{V}_0 d\rho \nabla \cdot \rho' = \chi d\rho. \tag{1}$$

We will use χ to denote the linity extended from the vector linity χ which is given by (1). Invariance has to do with relations which remain unchanged in form, in some defined sense or senses, when ρ' is taken as the independent variable instead of ρ. Suppose h is a scalar function of ρ. Then it is also a function of ρ'. Then

$$dh = -\mathrm{V}_0 d\rho \nabla h = -\mathrm{V}_0 d\rho' \nabla' h, \tag{2}$$

where ∇' is $\sum_{c=1}^{n} i_c \mathrm{D}_{x_c'}$. This is a relation of the kind just mentioned and we want to systematise the treatment of such relations. Anybody who reads my paper of 1892 will see how it came about that it should seem a very natural course to use the theorems of integration of § 8 above for the purpose.

We scarcely need to analyse the details of the strain after what has already been said. The ordinary notions of curl, $\mathrm{V}_2\nabla\sigma$, of an intensity, that is a covariant vector, σ, are given by (10) of § 8. The ordinary notions of convergence of a vector flux, $k\tau$, are given by (11) of § 8. [$k\tau$ is a vector flux; τ is a contravariant vector.] The ordinary notions of the half curl, $\frac{1}{2}\mathrm{V}_2\nabla\sigma$, representing the rate of rotation of a fluid whose velocity is σ, are just the same in n dimensions as in three [see (10) of § 4 and what has been said above about infinitesimal rotations]. But more:—it cannot surprise

* 'Phil. Trans.,' A, 1892, pp. 685–780.

anybody who has seen fluxes and intensities used as a mathematical method that there is no change in the same uses of the same expressions (curl, convergence, etc.) when we pass to an n-manifold of the most general kind. $V_2\nabla\sigma$ and $V_0\nabla(k\tau)$ will have just the same significance in the manifold as in the original Euclidean space; but this is anticipating.

Synopsis of Notation.

	Covariant.	Contravariant.
(1) Vector	σ	τ
(2) Hypervector	v	u
(3) Multenion	s	r
(4) Multenion linity	ξ	ξ^{-1}

We have given an instance of an invariantive statement in (2). Next, consider the integrals (10) and (7) of § 8, which we here reproduce,

$$\left.\begin{aligned} \int V_0\sigma d\rho &= \iint V_0 d\dot{\rho}_2 V_2\nabla\sigma \\ \int^{(b)}\!\!\int V_0 v d\dot{\rho}_b &= \int^{(b+1)}\!\!\int V_0 d\dot{\rho}_{b+1} V_{b+1}\nabla v \end{aligned}\right\}. \tag{3}$$

When ρ' is the independent variable, precisely these same integrals are to be expressed in the same form, thus :—

$$\left.\begin{aligned} \int V_0\sigma' d\rho' &= \iint V_0 d\dot{\rho}_2' V_2\Delta'\sigma' \\ \int^{(b)}\!\!\int V_0 v' d\dot{\rho}_b' &= \int^{(b+1)}\!\!\int V_0 d\rho_{b+1}' V_{b+1}\Delta' v' \end{aligned}\right\}. \tag{4}$$

Now, of course, we are compelled to identify our present $d\rho$ and $d\rho'$ with the $d\rho$ and $d\rho'$ of (1), so that $d\rho' = \chi d\rho$. Again, $d\dot{\rho}_b$ meant $V_b d\rho_1 d\rho_2 \ldots d\rho_b$ where each $d\rho_c$ again must be identified with the $d\rho$ of (1), and therefore the corresponding $d\rho'$ with the $d\rho'$ of (1). Hence

$$d\dot{\rho}' = \chi d\rho, \qquad d\dot{\rho}_b' = \chi d\dot{\rho}_b, \tag{5}$$

by the properties of extended vector linities.

When we say that the integrals (4) are the same as the integrals (3), we assert that each of the four elements of integration in (3) is equal to the corresponding element in (4). In other words, we assert the existence of four invariants,

$$\left.\begin{aligned} V_0\sigma d\rho &= V_0\sigma' d\rho', & V_0 d\dot{\rho}_2 V_2\nabla\sigma &= V_0 d\rho_2' V_2\nabla'\sigma' \\ V_0 v d\dot{\rho}_b &= V_0 v' d\dot{\rho}_b', & V_0 d\dot{\rho}_{b+1} V_{b+1}\nabla v &= V_0 d\dot{\rho}_{b+1}' V_{b+1}\nabla' v' \end{aligned}\right\}. \tag{6}$$

From (5) and the fundamental property of the conjugate of χ these at once give

$$\left.\begin{aligned} \sigma' &= \chi'^{-1}\sigma, & V_2\nabla'\sigma' &= \chi'^{-1} V_2\nabla\sigma \\ v' &= \chi'^{-1} v, & V_{b+1}\nabla' v' &= \chi'^{-1} V_{b+1}\nabla v \end{aligned}\right\}. \tag{7}$$

(5) and (7) suggest multenions of types r, s, such that

$$r^{\backslash} = \chi r, \qquad s^{\backslash} = \chi'^{-1} s, \tag{8}$$

r and s are multenion functions of ρ, and $r^{\backslash}$ and $s^{\backslash}$ are associated functions (say of $\rho^{\backslash}$) which produce the same integrals when $\rho^{\backslash}$ is taken as the independent variable in place of ρ [see the synopsis above]. When (8) gives the rule of the association of the functions,

r and s are said to be contravariant and covariant respectively.

It is well to place here the obvious consequence of (8)

$$\mathrm{V}_0 rs = \mathrm{V}_0 r^{\backslash} s^{\backslash}, \tag{9}$$

that is, $\mathrm{V}_0 pq$ is always an invariant if one of the two, p, q, is covariant and the other contravariant.

By exactly similar reasoning (a little complicated by the presence of k), we may treat of the equations (11) and (8) of § 8. For the sake of saving space, I merely state the results, k being given by (9) of § 8, and τ, u, $d\varsigma$ (including as particular cases $d\alpha$ and db) being as in (11) and (8) of § 8, the following are covariant:—

$$k\acute{v}, \qquad kdb, \qquad k\alpha, \qquad kd\varsigma_a$$

and the following are contravariant

$$kdb, \qquad \tau, \qquad u, \qquad k^{-1}\mathrm{V}_0\nabla(k\tau), \qquad k^{-1}\mathrm{V}_{a-1}\nabla(ku),$$

kdb is both covariant and contravariant. This merely means that it is invariant ($kdb = k^{\backslash}db^{\backslash}$). Remembering that $\chi h = h = \chi'^{-1}h$ when h is a scalar, (8) shows at once that, for a scalar covariance and contravariance, both mean precisely the same as invariance.

The persistent manner in which k thrusts itself into these results tempts one to wonder whether covariant and contravariant terminology is superior to intensity and flux terminology. After some experience of using both, I am inclined to believe that the accepted covariant and contravariant standard scheme is superior to the other.

It is not difficult to show that $k\acute{v}$ satisfies the test of (8) for covariance, that is

$$k^{\backslash}\acute{v} = \chi'^{-1}(k\acute{v}).$$

If in (2) and (3) of § 6 we read χ in place of ϕ, we get theorems closely connected with multiplication and composition in the absolute differential calculus. Let v_a, v_b be covariant multenions of homogeneities, a, b. Then

$$v_{a+b} = \mathrm{V}_{a+b} v_a v_b, \tag{9}$$

is covariant (multiplication). If u_a, u_b, are similar contravariants then

$$u_{a+b} = \mathrm{V}_{a+b} u_a u_b, \tag{10}$$

is contravariant (multiplication). If u is contravariant (homogeneity a) and v is covariant (homogeneity b), then

$$u_{a-b} = \mathrm{V}_{a-b}uv, \tag{11}$$

is contravariant (composition) when $a > b$. Also

$$v_{b-a} = \mathrm{V}_{b-a}uv, \tag{12}$$

is covariant (composition) when $a < b$. Lastly, when $a = b$, then each is invariant, which is a special case of (9). A special case arises when we put the larger of the two, a, b equal to n.

ξ, any multenion linity, is said to be covariant when it produces a covariant multenion from a contravariant, thus

$$s = \xi r, \qquad s^{\backprime} = \xi^{\backprime} r^{\backprime}. \tag{13}$$

Conversely since then

$$r = \xi^{-1} s, \qquad r^{\backprime} = \xi^{\backprime -1} s^{\backprime}, \tag{14}$$

ξ^{-1} is said to be contravariant. From (8) we at once get

$$\xi^{\backprime} = \chi'^{-1} \xi \chi^{\backprime -1}, \qquad \xi^{\backprime -1} = \chi \xi^{-1} \chi'. \tag{15}$$

From this, when ξ is covariant, its conjugate ξ' is also covariant, and, therefore, both the self-conjugate part and the skew part of ξ are covariant. Similarly for ξ^{-1} and contravariance. Also, when ξ is self-conjugate, $\xi^{\backprime}$ is self-conjugate. If ξ is a covariant vector linity, the extended linity is also covariant. Also, in this case, the rotation-${}_n\mathrm{V}_2$ (see (10) of § 4) $\frac{1}{2}\mathrm{V}_2\zeta\xi\zeta$ is covariant. To prove this last, we have to show that

$$\chi'\mathrm{V}_2\zeta\xi^{\backprime}\zeta = \mathrm{V}_2\zeta\xi\zeta.$$

Now

$$\begin{aligned} \chi'\mathrm{V}_2\zeta\xi^{\backprime}\zeta &= \mathrm{V}_2\chi'\zeta\chi'\xi^{\backprime}\zeta \quad && [(2)\ \S\ 6] \\ &= \mathrm{V}_2\zeta\chi'\xi^{\backprime}\chi\zeta && [(9)\ \S\ 5] \\ &= \mathrm{V}_2\zeta\xi\zeta && [(15)\ \text{above}], \end{aligned}$$

as required.

Let $\sigma_1\ \sigma_2\ \ldots,\ \sigma_n$ be n independent covariant vectors. [They might be taken as ∇h_1, $\nabla h_2\ \ldots\ \nabla h_n$ where the Jacobian of the quantities h does not vanish, that is, where $\mathrm{V}_n(\nabla h)^{(n)} \neq 0$.] From the relations

$$\left.\begin{aligned} 1 &= \mathrm{V}_0\sigma_1\bar{\sigma}_1 = \mathrm{V}_0\sigma_2\bar{\sigma}_2 = \ldots \\ 0 &= \mathrm{V}_0\sigma_1\bar{\sigma}_2 = \mathrm{V}_0\sigma_1\bar{\sigma}_3 = \ldots \end{aligned}\right\}, \tag{16}$$

it at once follows that $\bar{\sigma}_1$, $\bar{\sigma}_2$, … $\bar{\sigma}_n$ are n independent contravariant vectors, and σ_c, $\bar{\sigma}_c$ might just as properly have been denoted by $\bar{\tau}_c$, τ_c. Any covariant or contravariant vector can be expressed at once in terms of these and invariants. Also, from the 2^n vectoriums formed from each set,

any covariant or contravariant multenion may be similarly expressed. Thus the expressions for σ, τ, v, u are [see (2) of § 5]

$$\left.\begin{array}{ll} \sigma = \sum_{c=1}^{n} \sigma_c \mathrm{V}_0 \sigma \bar{\sigma}_c, & \tau = \sum_{c=1}^{n} \bar{\sigma}_c \mathrm{V}_0 \tau \sigma_c, \\ v = \Sigma \mathrm{V}_b \sigma^{(b)} \,.\, \mathrm{V}_0 v \overline{\mathrm{V}_b \sigma^{(b)}}, & u = \Sigma \overline{\mathrm{V}_a \sigma^{(a)}} \,.\, \mathrm{V}_0 u \mathrm{V}_a \sigma^{(a)} \end{array}\right\}. \tag{17}$$

§ 10. *Differentiation.*—Our integration theorems have suggested certain systematisations, now to be given in the use of ∇. We have

$$\nabla w = \mathrm{V}_{c+1} \nabla w + \mathrm{V}_{c-1} \nabla w.$$

If w is covariant, one part on the right, viz., the first, is covariant and the other not. A similar statement applies to contravariance. We cannot, as seems desirable, resolve the symbolic vector ∇ into two parts, to which each of these two discordant parts on the right belong. But we can so resolve the symbolic *linity* $\nabla(\ )$. The first part of the linity, connected with $\mathrm{V}_{c+1}\nabla w$ produces a part of ∇w of homogeneity one unit *higher* than that of w, and may be thought of as the *ascending* derivative linity, and will be denoted by H. The other part, L, produces a *lower* homogeneity, and may be thought of as the *descending* derivative. Thus, formally, we define the linities H and L thus

$$\left.\begin{array}{ll} \nabla(\ ) = \mathrm{H} + \mathrm{L} & \\ \text{where} \quad \mathrm{H} = \sum_{c=0}^{n-1} \mathrm{V}_{c+1} \,.\, \nabla \mathrm{V}_c(\ ), & \mathrm{L} = \sum_{c=1}^{n} \mathrm{V}_{c-1} \,.\, \nabla \mathrm{V}_c(\ ) \end{array}\right\}. \tag{1}$$

$$\left.\begin{array}{lll} \text{Otherwise} & \nabla w = \mathrm{H}w + \mathrm{L}w & \\ \text{where} & \mathrm{H}w = \mathrm{V}_{c+1}\nabla w, & \mathrm{L}w = \mathrm{V}_{c-1}\nabla w \end{array}\right\}. \tag{2}$$

There are two, wholly separate, aspects of H and L as operators. Each is both a linear operator and a differential operator. In (2) the subject of these two operations is the one symbol, w, but in (4), below (where H′ is the conjugate of H), in

$$\phi_g \mathrm{H}_g' d\rho_{b+1},$$

we have a case where H′, as linear operator, affects the immediately succeeding symbol, $d\rho_{b+1}$, while as differential operator it affects ϕ, and this is indicated by suffixes in the usual manner. This example illustrates some of the facilities of operation provided by H.

In considering the fundamental properties of H and L, think of replacing

$$\begin{array}{lllll} & \nabla, & \nabla w, & \mathrm{V}_{c+1}\nabla w, & \mathrm{V}_{c-1}\nabla w \\ \text{by} & \alpha, & \alpha w, & \mathrm{V}_{c+1}\alpha w, & \mathrm{V}_{c-1}\alpha w, \end{array}$$

where α is an ordinary vector instead of a symbolic vector. Let H′, L′

be the conjugates of H, L as usual. From the fundamental relation, $V_0 p\phi q = V_0 q\phi' p$, we at once have

$$H'w = V_{c-1}w\nabla, \qquad L'w = V_{c+1}w\nabla. \tag{3}$$

Thus the integration theorems (1) and (4) of § 8 read

$$\int^{(b)}\int \phi d\dot{\rho}_b = \int^{(b+1)}\int \phi_g H_g' \, d\dot{\rho}_{b+1}, \tag{4}$$

$$\int^{(n-a)}\int \psi d\varsigma_a = \int^{(n-a+1)}\int \psi_g L'_g d\varsigma_{a-1}, \tag{5}$$

and (7) and (8) of § 8 read

$$\int^{(b)}\int V_0 v d\dot{\rho}_b = \int^{(b+1)}\int V_0 d\dot{\rho}_{b+1} Hv, \tag{6}$$

$$\int^{(n-a)}\int ukd\varsigma_a = \int^{(n-a+1)}\int V_0 d\varsigma_{a-1} L(ku). \tag{7}$$

Further we have

$$H^2 = 0 = L^2, \tag{8}$$

and by taking conjugates we have

$$H'^2 = 0 = L'^2.$$

From (8) consider the various powers $\nabla^2(\ )$, $\nabla^3(\ )$ of $\nabla(\ )$. Clearly we get

$$\left.\begin{aligned} \nabla^2(\ ) &= HL + LH \\ \nabla^3(\ ) &= HLH + LHL, \text{ etc.} \end{aligned}\right\}, \tag{9}$$

and in (9) we may take conjugates [noting that, since the conjugate of $\nabla(\ )$ is $(\ )\nabla$, the conjugate of $\nabla^2(\ )$ is $(\ )\nabla^2$, etc.].

Notice that H′ *lowers* the homogeneity, while L′ *heightens* it, so that we have the following Table:—

	Heightens homogeneity.	Lowers homogeneity.
Prefixes ∇	H	L
Postfixes ∇	L′	H′

Either of the two, H and L, can be expressed in terms of the other thus

$$Lq = H(q\dot{v}) \,.\, \dot{v}^{-1}, \qquad Hq = L(q\dot{v}) \,.\, \dot{v}^{-1}. \tag{10}$$

Certain identities satisfied by H and L are given by the theorems of integration. As an example, $\int d\rho' = 0$ for a closed curve. This can be expressed in terms of ρ thus:

$$0 = \int \chi d\rho = \iint \chi_g V_1 d\dot{\rho}_2 \nabla_g.$$

Since the whole tract may be taken as the element $d\dot{\rho}_2$ we get

$$\chi_g V_1 d\dot{\rho}_2 \nabla_g = 0. \tag{11}$$

If we operate by $V_0\gamma(\ )$ we get

$$V_0 d\dot{\rho}_2 V_2 \nabla\chi'\gamma,$$

or since $d\dot{\rho}_2$ is arbitrary,

$$V_2\nabla\chi'\gamma = 0. \tag{12}$$

Both (11) and (12) are simple enough to be established directly from the form of χ, namely, $\chi\gamma = -V_0\gamma\nabla \cdot \rho^{\backprime}$, but the method of deriving them, just given, suggests how to prove other results not nearly so easy to establish directly. Instead of $\int d\rho^{\backprime} = 0$ we may take either of

$$\int^{(b)}\int d\dot{\rho}_b^{\backprime} = 0, \qquad \int^{(n-a)}\int ds_a^{\backprime} = 0,$$

and modify them in the same way. Thus from the first, $0 = \int^{(b+1)}\int \chi_g H_g' d\dot{\rho}_{b+1}$. From this and by taking conjugates we get

$$\chi_g H_g' q = 0, \qquad H_g\chi_g' q = 0, \tag{13}$$

where q is an arbitrary multenion. If we put $q = w$ (13) becomes

$$\chi_g V_{c-1} w\nabla_g = 0, \qquad V_{c+1}\nabla_g\chi_g' w = 0. \tag{14}$$

Similarly we have

$$0 = \int^{(n-a)}\int ds_a^{\backprime} = \int^{(n-a)}\int m\chi^{1-1} ds_a.$$

m is here $k/k^{\backprime}$, *i.e.*, it is the Jacobian of $\rho^{\backprime}$, or by (11) of § 7

$$m = n' V_0 V_n \nabla^{(n)} V_n \epsilon^{(n)}.$$

Thus

$$0 = \int^{(n-a+1)}\int (m\chi^{1-1})_g L_g' ds_{a-1}.$$

Hence, and by taking conjugates

$$(m\chi'^{-1})_g L_g' q = 0, \qquad L_g(m\chi^{-1})_g q = 0. \tag{15}$$

Putting $q = w$ we get

$$(m\chi'^{-1})_g V_{c+1} w\nabla_g = 0, \qquad V_{c-1}\nabla_g (m\chi^{-1})_g w = 0. \tag{16}$$

Let us now find expressions explicitly giving $\nabla^{\backprime}$, $H^{\backprime}$, $L^{\backprime}$ in terms involving ∇, H, L. By (2), § (9), ∇ is a symbolic covariant vector, so that

$$\nabla^{\backprime} = \chi'^{-1}\nabla, \tag{17}$$

where on the right the differentiations are not to affect χ'^{-1}.

For H we have that Hs is covariant when s is. Hence

$$H^{\backprime}s^{\backprime} = \chi'^{-1}Hs = \chi'^{-1}H(\chi's').$$

Now, by the second of (13) in this expression we may show the differentiations of H explicitly as affecting or as not affecting the immediately following χ^1, because the terms implied by those differentiations are identically zero. Hence we may write

$$H^{\backprime} = \chi'^{-1} \cdot H \cdot \chi', \tag{18}$$

bearing this permissibility of interpretation in mind. The relations between

H' and χ, on the one hand, and H' and X^{-1}, on the other, are precisely the same, so that we may write

$$H = \chi' \,.\, H' \,.\, \chi'^{-1}, \tag{19}$$

with permissibility of interpretation as in (18).

Similarly from (18) and (19) by means of (5) of § 7

$$L' = (m^{-1}\chi) \,.\, L \,.\, (m\chi^{-1}), \tag{20}$$

$$L = (m\chi^{-1}) \,.\, L' \,.\, (m^{-1}\chi), \tag{21}$$

with the same permissible interpretation as to the incidence of the application of the differentiations of L and L' on the right; they may be supposed to affect the immediately following $(m\chi^{-1})$ and $(m^{-1}\chi)$ or not, at our pleasure.

A New Form of Wehnelt Interrupter.

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[Plates 3 and 4.]

1. *Introduction.*

The original form of Wehnelt Interrupter has many disadvantages. Unless special precautions for cooling the apparatus are taken, the solution soon boils, and the interrupter ceases to work. The large current density causes rapid disintegration of the platinum wire, and there is considerable expense in renewing it. The interrupter cannot be used with alternating currents owing to the melting of the wire when it is the cathode.

Sulphuric acid has a fairly large electrical conductivity, compared with other electrolytes, and the mean value of the current density at the wire electrode is large. There is, as a result, considerable heat developed in the volume of the acid. If a high resistance electrolyte is substituted, the current density at the platinum wire is much smaller, and the heating effect is reduced. The fumes arising from the acid, and also the spraying, when the interrupter is in action, are objectionable.

Although electrolytic interrupters are not widely used in the laboratory, they are in considerable demand for X-ray work, because of the heavy disruptive discharges which are obtained in the secondary of a coil operated